Coulomb's Law :-

Coulomb's Law states that the force between two very small objects separated in a vacuum or free space by a distance which is large compared to their size is proportional to the charge on each and inversely proportional to the square of the distance between them. i.e.,

$$\boxed{F = \frac{KQ_1Q_2}{R^2}} \text{ Newtons.}$$

where Q_1 & Q_2 are the +ve or −ve quantities of charge

$R \rightarrow$ separation between the two charges.

$K \rightarrow$ proportionality constant.

→ Due to the force is inversely proportional to the square of the distance, so this (Coulomb's) Law also called '<u>Inverse Square Law</u>'.

→ Since the constant of proportionality K is written as

$$\boxed{K = \frac{1}{4\pi\varepsilon_0}}$$

where $\varepsilon_0 \rightarrow$ permittivity of free space.

$$\varepsilon_0 = 8.854 \times 10^{-12} = \frac{1}{36\pi} * 10^{-9} \text{ F/m.}$$

Then the Coulomb's Law becomes

$$\boxed{F = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^2}}$$

→ The Coulomb is an extremly large unit of charge.

→ for the smallest known quantity of charge that of the electron or proton is 1.602×10^{-19} C and hence a negative charge of one Coulomb represents about 6×10^{18} electrons.

Ex:— What is the force between two charges of one Coulomb each, separated by one meter?

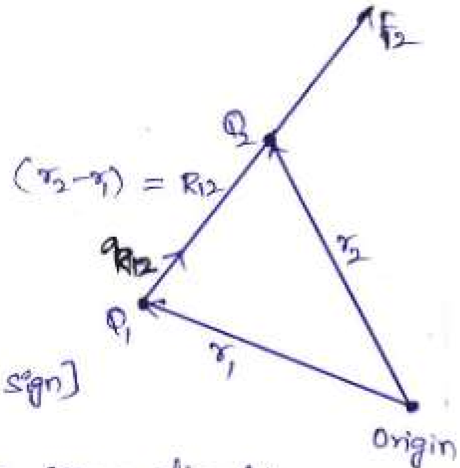
Sol:— $Q_1 = 1C$, $Q_2 = 1C$, $R = 1$ mts.

$\therefore$ from Coulomb's Law, $f = \dfrac{Q_1 Q_2}{4\pi\epsilon_0 R^2}$

$$f = \frac{1\times1}{4\pi\cdot\epsilon_0\cdot(1)^2} = \frac{1}{4\pi * \frac{1}{36\pi} * 10^{-9}} = 9\times10^{9} \text{ Newtons.}$$

→ Since, the force is a vector Quantity having magnitude & Direction both. So we have to represent the force in vector form.

→ In vector form, the force acts along the line joining the two charges and is repulsive if the charges are like in sign and attractive if they are of opposite sign.



→ Let two charges Q_1 and Q_2 which are located at vector distances of r_1 and r_2 respectively from the origin and assume Q_1 and Q_2 charges have like signs [same sign] then the vector force f_2 on Q_2 is in the same direction as the vector R_{12}. Here the vector $R_{12} = (r_2 - r_1)$ represents the directed line segment from Q_1 to Q_2 as shown in figure. Then the vector f_2 is the force on Q_2 due to Q_1 is given by

**
$$\boxed{F_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}\, a_{R12}}$$

Where a_{R12} = a unit vector in the direction of R_{12} (or)

$$\boxed{a_{R12} = \frac{\bar{R}_{12}}{|R_{12}|} = \frac{r_2 - r_1}{|r_2 - r_1|}} = \frac{r_2 - r_1}{R} \longrightarrow (a)$$

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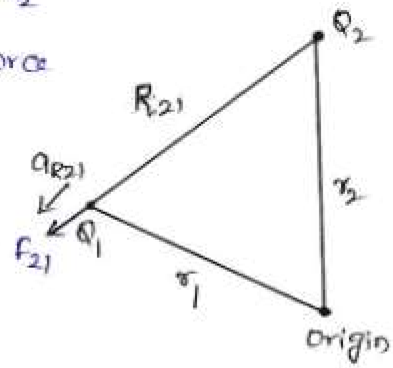
→ The vector F_{21} is force on Q_1 due to Q_2 where Q_1 and Q_2 have the same sign. so the force is Repulsive force. Then the vector form of Coulomb's Law is



$$\boxed{F_{21} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{21}^2} \cdot a_{R21}}$$

Where a_{R21} = a unit vector in the direction of R_{21}, or

$$\boxed{a_{R21} = \frac{\overline{R_{21}}}{|R_{21}|} = \frac{r_1 - r_2}{|r_1 - r_2|}} \longrightarrow (b)$$

By Comparing eqns (a) & (b),

$$\boxed{a_{R12} = -a_{R21}}$$

i.e., only direction is opposite. so the force is same in magnitude but the opposite in direction.

→ The force expressed by Coulomb's Law is a mutual force i.e., for each of the two charges experiences a force of the same magnitude, although of opposite direction.

→ We can write the expression for force from Coulomb's Law is

$$\boxed{F_{12} = -F_{21} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2} \cdot a_{R12} = \frac{-Q_1 Q_2}{4\pi\varepsilon_0 |R_{21}|^2} \cdot a_{R21}}$$

*→ Coulomb's Law is linear, if we multiply Q_1 by a factor n, the force on Q_2 is also multiplied by the same factor n.

→ If more than two charges are present, then the force on a particular charge is the sum of the forces on that charge due to each of the other charges acting alone.

Ex:— Consider a charge of 3×10^{-4} C at P(1,2,3) and a charge of -10^{-4} C at Q(2,0,5) in a vacuum. Then find out the force acting on charge at 'Q' due to a charge at 'P' → ?

Ans:

$$Q_1 = 3\times10^{-4} C \quad Q_2 = -10^{-4} C$$

$$R_{12} = r_2 - r_1 = (2-1)a_x + (0-2)a_y + (5-3)a_z$$

$$= a_x - 2a_y + 2a_z.$$

$$a_{R12} = \frac{R_{12}}{|R_{12}|} = \frac{a_x - 2a_y + 2a_z}{3}$$

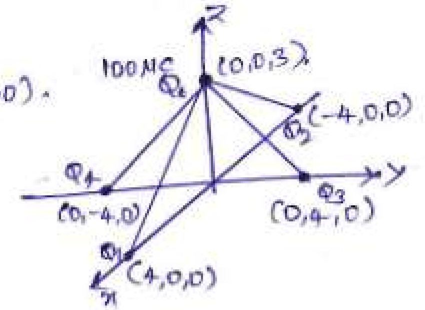
$$\therefore F_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 |R_{12}^2|} \cdot \frac{R_{12}}{|R_{12}|} = \frac{3\times10^{-4} * -10^{-4}\ [a_x - 2a_y + 2a_z]}{4\pi * \frac{1}{36\pi * 10^{9}} * (3)^3}$$

$$= -10^{-8} * 10^{9}\ [a_x - 2a_y + 2a_z] \quad \text{Newtons.}$$

$$\underline{F_{12} = -10a_x + 20a_y - 20a_z \quad N.}$$

Prob:— Find the force on a 100μC at (0,0,3) m if four like charges of 20μC are located on the 'x' and 'y' axes at ±4 m?

Ans: Let the four charges, which are at Q_1 at (4,0,0), Q_2 at (−4,0,0), Q_3 at (0,4,0) & Q_4 at (0,−4,0).



Q_1 (4,0,0) Q_t (0,0,3)

$R_{1t} = -4a_x + 3a_z.$

$|R_{1t}| = 5$

$$F_{1t} = \frac{Q_1 Q_t}{4\pi\varepsilon_0 |R_{1t}^2|} \cdot \frac{R_{1t}}{|R_{1t}|}$$

$$= \frac{(20\times10^{-6})(10^{-4})}{4\pi * \frac{1}{36\pi * 10^{9}} *} \frac{[4a_x - 3a_z]}{(5)^3} = \frac{20 * 10^{-10} * 10^{9}}{1/9} * \frac{[4a_x + 3a_z]}{125}$$

$$= \frac{180 * 10^{-1}}{125}[4a_x + 3a_z] = \frac{18}{125}[4a_x + 3a_z].$$

$$F_{2t} = \frac{Q_2 Q_t}{4\pi\varepsilon_0} \frac{R_{2t}}{|R_{2t}|^3} \Rightarrow F_{2t} = 1.73[4a_x + 3a_z]$$

$$F_{3t} = \frac{Q_3 Q_t}{4\pi\varepsilon_0} \cdot \frac{R_{3t}}{|R_{3t}|^3} = \frac{20 * 10^{-10}}{\frac{1}{9\times10^{9}}} \frac{[-4a_y + 3a_z]}{(5)^3}$$

Q_3 (0,4,0) Q_t (0,0,3)

$$F_{3t} = \frac{Q_3 Q_t}{4\pi\varepsilon_0} \cdot \frac{R_{3t}}{|R_{3t}|^3}$$

$R_{3t} = -4a_y + 3a_z$

$|R_{3t}| = 5$

$$F_{3t} = \frac{18}{125}[-4a_y + 3a_z]$$

Similarly

$$F_{4t} = \frac{18}{125}[4a_y + 3a_z]$$

∴ force on Q_t due to all the four charges is

$$F = F_{1t} + F_{2t} + F_{3t} + F_{4t}$$

$$= \frac{18}{125}[\cancel{-4a_x} + 3a_z] + \frac{18}{125}[\cancel{4a_x} + 3a_z] + \frac{18}{125}[\cancel{-4a_y} + 3a_z] + \frac{18}{125}[\cancel{4a_y} + 3a_z]$$

$$F = \frac{18}{125}[12a_z] = 1.73\, a_z \text{ Newtons.}$$

→ If we have more than two point charges, we can use the principle of superposition to determine the force on a particular charge. The principle states that if there are 'N' charges $Q_1, Q_2, \ldots, Q_N$ located, respectively, at points with position vectors $r_1, r_2, \ldots r_N$, the resultant force 'F' on a charge 'Q' located at point 'r' is the vector sum of the force exerted on 'Q' by each of the charges $Q_1, Q_2, \ldots, Q_N$. Hence:

$$F = \frac{QQ_1(r-r_1)}{4\pi\varepsilon_0|r-r_1|^3} + \frac{QQ_2(r-r_2)}{4\pi\varepsilon_0|r-r_2|^3} + \cdots + \frac{QQ_N(r-r_N)}{4\pi\varepsilon_0|r-r_N|^3}$$

$$\therefore \boxed{F = \frac{Q}{4\pi\varepsilon_0}\sum_{k=1}^{N}\frac{Q_k\cdot(r-r_k)}{|r-r_k|^3}}$$

→ **field:**— The Region in which at every point of region there is a corresponding value of some physical function. That region is called "field".

Electric Field Intensity :-

→ Electric field intensity is defined as the 'force per unit charge'.

→ If we consider one charge fixed in position, say Q_1 and move a second charge slowly around the fixed charge. We observe that there exists everywhere a force on this second charge. This second charge displaying the existence of a force field. Let this second charge a test charge 'Q_t'. From the Coulomb's Law, the force on the test charge is given by

$$\boxed{F_t = \frac{Q_1 Q_t}{4\pi\epsilon_0 R_{1t}^2} a_{R1t}}$$

We can write the above expression as

$$\frac{F_{1t}}{Q_t} = \frac{Q_1}{4\pi\epsilon_0 R_{1t}^2} a_{R1t}$$

→ The quantity on the right hand side of above eqn is a function only of 'Q_1' and the directed line segment 'Q_1' to the position of the test charge. This describes a vector field and is called the 'Electric field Intensity'.

→ We define the electric field intensity as "the vector force on a unit positive test charge".

→ So, the Electric field Intensity expression due to single point charge 'Q_1' in a vacuum is **

$$\boxed{E = \frac{F_{1t}}{Q_t} = \frac{Q_1}{4\pi\epsilon_0 R_{1t}^2} a_{R1t}}$$

Where a_{R1t} is the unit vector directed from Q_1 to the point where we are desired to find the electric field intensity.

→ Let us arbitrarily locate 'Q_1' at the center of a spherical co-ordinate system. The unit vector a_R then becomes the radial unit vector a_r, and R is r. Hence

$$\boxed{E = \frac{Q_1}{4\pi\varepsilon_0 r^2} \cdot a_r}$$

(or) $$\boxed{E_r = \frac{Q_1}{4\pi\varepsilon_0 r^2}}$$

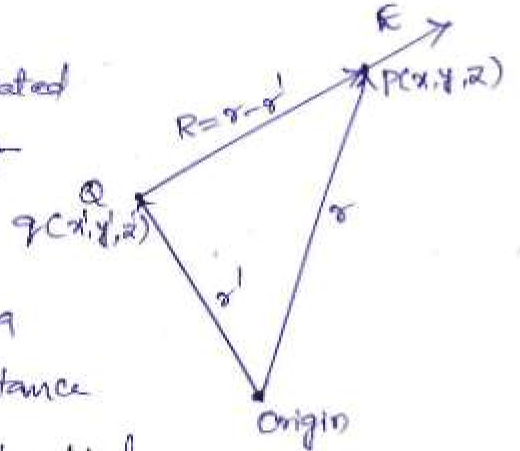
→ By writing these expressions in cartesian coordinates for a charge 'Q_1' at the origin, we have

$$R = r = x a_x + y a_y + z a_z,\quad a_R = a_r = (x a_x + y a_y + z a_z)/\sqrt{x^2+y^2+z^2},$$

and then $$\boxed{E = \frac{Q_1}{4\pi\varepsilon_0 (x^2+y^2+z^2)}\left(\frac{x}{\sqrt{x^2+y^2+z^2}} a_x + \frac{y}{\sqrt{x^2+y^2+z^2}} a_y + \frac{z}{\sqrt{x^2+y^2+z^2}} a_z\right)}$$

→ If we consider a charge which is not at the origin of our coordinate system, the field no longer possess a spherical symmetry, we might use cartesian coordinates.

→ Let us consider a charge 'Q' is located at a point $q(x', y', z')$ which is at a distance 'r'' from the origin. We desired to find the field at a point $P(x, y, z)$ which is at a distance of 'r' from the origin. We have to find the field at 'P' due to a charge 'Q'. The spacing between these two points can have $R = r - r'$ vectorically. So the field is along the direction of $(r - r')$.



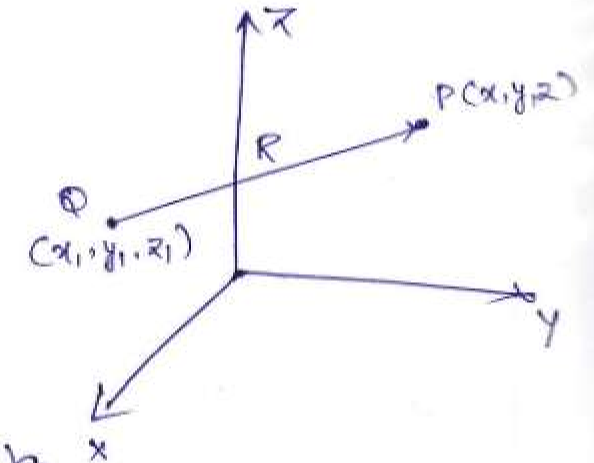
Here $r = x a_x + y a_y + z a_z$, $r' = x' a_x + y' a_y + z' a_z$

then

$$E(r) = \frac{Q}{4\pi\varepsilon_0 |r-r'|^2} \frac{(r-r')}{|r-r'|} = \frac{Q\cdot(r-r')}{4\pi\varepsilon_0 |r-r'|^3}$$

$$\boxed{E(r) = \frac{Q\left[(x-x')a_x + (y-y')a_y + (z-z')a_z\right]}{4\pi\varepsilon_0 \left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{3/2}}}$$

→ Expression for the Electric field at P due to a point charge Q at (x_1,y_1,z_1). Repeat with the charge placed at the origin?



from the figure,

$R = (x-x_1)a_x + (y-y_1)a_y + (z-z_1)a_z$

then $E = \frac{Q}{4\pi\varepsilon_0 R^2} a_R$

$$= \frac{Q}{4\pi\varepsilon_0} \frac{(x-x_1)a_x + (y-y_1)a_y + (z-z_1)a_z}{\left[(x-x_1)^2+(y-y_1)^2+(z-z_1)^2\right]^{3/2}}$$

When the charge is at the origin,

$$E = \frac{Q}{4\pi\varepsilon_0} \frac{(xa_x + ya_y + za_z)}{(x^2+y^2+z^2)^{3/2}}$$

This expression fails to show the symmetry of the field.

In spherical coordinates with Q at the origin,

$$\boxed{E = \frac{Q}{4\pi\varepsilon_0 r^2} \cdot a_r}$$

→ Incase of Electric field due to two point charges:

Let us assume two point charges Q_1 & Q_2, which are located r_1 and r_2 distances respectively from origin. we desired to find the field at a particular point due to these two point charges.

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Since the Coulomb forces are linear. The total Electric field due to two point charges is the sum of fields due to individual charges acting at a particular point where we desired to find the field.

Therefore,

$$\boxed{E(r) = \frac{Q_1}{4\pi\epsilon_0 |r-r_1|^2} a_1 + \frac{Q_2}{4\pi\epsilon_0 |r-r_2|^2} \cdot a_2}$$

Where a_1 and a_2 are unit vectors in the direction of $(r-r_1)$ and $(r-r_2)$ respectively.

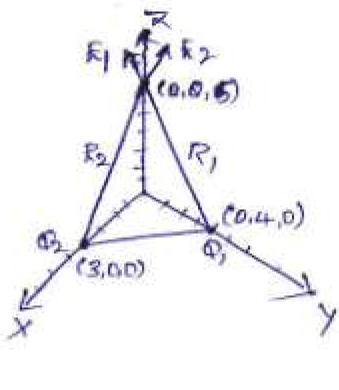
→ If we add more charges at other positions, the field due to 'n' point charges is

$$E(r) = \frac{Q_1}{4\pi\epsilon_0 |r-r_1|^2} \cdot a_1 + \frac{Q_2}{4\pi\epsilon_0 |r-r_2|^2} a_2 + \dots\dots + \frac{Q_n}{4\pi\epsilon_0 |r-r_n|^2} \cdot a_n$$

Simply we can write.

$$\boxed{E(r) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0 |r-r_m|^2} a_m}$$

Prob:— Find E at $(0,0,5)$ m due to $Q_1 = 0.35\mu C$ at $(0,4,0)$ m and $Q_2 = -0.55\mu C$ at $(3,0,0)$ m. [See in figure]



Sol:—

$R_1 = [(0-0)a_x + (0-4)a_y + (5-0)a_z]$

$R_1 = -4a_y + 5a_z$

$R_2 = [(0-3)a_x + (0-0)a_y + (5-0)a_z] = -3a_x + 5a_z$

$$E_1 = \frac{0.35\times10^{-6}}{4\pi(10^{-9}/36\pi)(41)} \frac{(-4a_y+5a_z)}{\sqrt{41}} = -48.0a_y + 60.0a_z \text{ V/m}$$

$$E_2 = \frac{-0.55\times10^{-6}}{4\pi(10^{-9}/36\pi)(34)} \frac{(-3a_x+5a_z)}{\sqrt{34}} = 74.9a_x - 124.9a_z \text{ V/m}$$

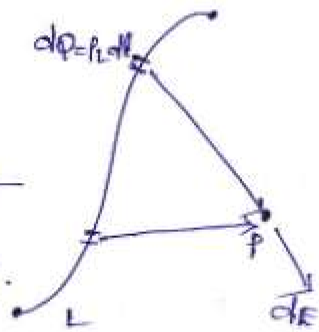
$E = E_1 + E_2 = 74.9a_x - 48.0a_y - 64.9a_z$ V/m //

Fields Due to Different Charge Distributions :—

Line charge :—

If charge Q is distributed over a (Curved) Line. for finding the field at any point due to a line charge, devide the entire length of line into equal infinitely small lengths (approximately a point charge) of charge. so we have to find the total field at a particular point by adding fields due to individual small charge elements. since each differential charge dQ along the line produces a differential electric field at a particular point P is

$$\boxed{dE = \frac{dQ}{4\pi\varepsilon_0 R^2} a_R}$$



→ The length of each individual small element is dl, and ρ_L is called 'Line charge Density'. which one is uniformly Distributed i.e., each small length of the line has equal line charge density ρ_L.

so the charge due to the individual small element of the line is

$$\boxed{dQ = \rho_L dl.} \quad \& \quad \boxed{Q = \int \rho_L dl}$$

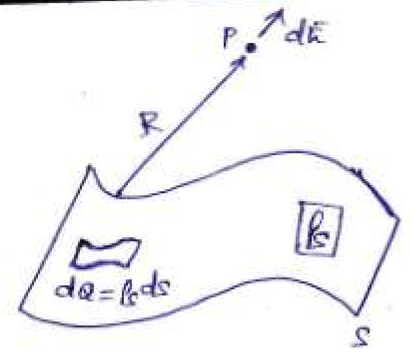
→ so the Electric field due to the individual small element is

$$dE = \frac{\rho_L dl}{4\pi\varepsilon_0 R^2} . a_R .$$

then the total field due to total length of line can be calculate by integrating the field due to individual small element through-out the length. The expression is

$$\boxed{E = \int_L \frac{\rho_L a_R}{4\pi\varepsilon_0 R^2} . dl}$$

Surface charge :—



Assume a sheet 's' of charge Q.

If we desired to find the field at any point due to this sheet, devide the entire Area of the sheet into small sheets (i.e. infinitely small in Area) and assume the surface charge density ρ_s (C/m²) is uniform i.e., the differential charge of each infinitely small area is same.

Assume the area of individual small sheets is ds so the differential charge due to individual small area is

$$\boxed{dQ = \rho_s \cdot ds}$$

Here ρ_s is constant (or uniform) for all sheets of small area.

→ The total charge due to all these individual small areas is

$$\boxed{Q = \int_s \rho_s \cdot ds} \quad \text{(surface charge)}.$$

→ If we desire to find the field at any point, first we have to find the field of individual small area at that point. Then we have to integrate that field over the entire area of sheet 's'.

field due to individual (infinitely) small area at a particular point is

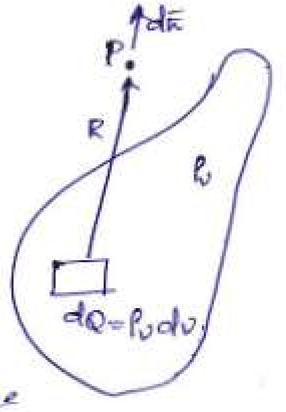
$$\boxed{dE = \frac{dQ}{4\pi\varepsilon_0 R^2} a_R}$$

→ Surface charge density ρ_s (C/m²) is uniform, so the total field due to entire Area is

$$** \quad \boxed{E = \int_s dE \, ds = \int_s \frac{\rho_s \cdot a_R}{4\pi\varepsilon_0 R^2} ds}$$

Volume Charge :

Assume volume of charge 'Q'. This charge 'Q' is uniformly spread over the entire volume 'v'. i.e., the volume charge density is uniform.



Let us assume the volume charge density is 'ρ_v', which is uniform. i.e., if we devide the entire volume into small parts of volume this individual parts has the same charge density. Assume the volume of individual small part is 'dv' and the total available charge is uniformly spread over the volume so the individual part charge is 'dQ'. so the individual part of the volume has the charge density of

$$\boxed{\rho_v = \frac{dQ}{dv}} \quad (C/m^3)$$

→ The total volumic charge can be found by integrating the above eqn over the volume i.e.,

$$\boxed{Q = \int_v \rho_v \, dv} \quad \text{Coulombs}$$

→ Each individual part has the charge of dQ, due to this field produced at any point is

$$dE = \frac{dQ}{4\pi\epsilon_0 R^2} a_R$$

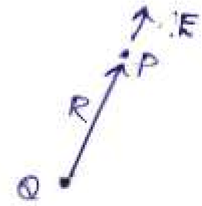
Since $dQ = \rho_v \cdot dv$, so $$\boxed{dE = \frac{\rho_v \cdot dv}{4\pi\epsilon_0 R^2} a_R}$$

→ Total field due to the entire volume is the sum of fields due to individual parts (or) total field can be found by integrating the field due to individual part.

**

$$\boxed{E = \int_V \frac{\rho_v a_R}{4\pi\epsilon_0 R^2} dv}$$

Point charge:—



field due to a point charge 'Q' at a distance 'R' from the point charge is

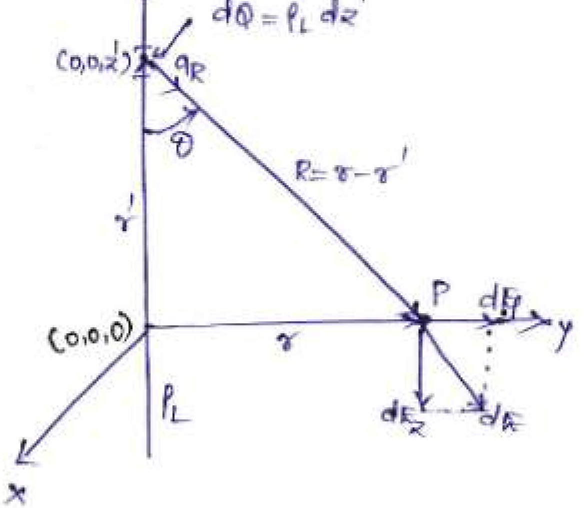
$$\boxed{E = \frac{Q}{4\pi\epsilon_0 R^2} a_R}$$ (spherical coordinates)

This is a spherically symmetric field.

Field of a Line charge:—

→ Assume a charged conductor of very small radius. It is convenient to treat the charge as a line charge of density ρ_L C/m.

Let us assume a straight line charge extending along the z-axis in a cylindrical coordinate system from $-\infty$ to ∞. We desire the Electric field intensity E at any and every point resulting from a uniform line charge density ρ_L.



→ symmetry should be considered to know the components of the field which are present and with which coordinates field is to be varied.

→ from the figure, as we move around the line charge i.e, varying 'φ' [Azimuth angle] while keeping 'ρ' and 'z' constant. The line charge appears the same from every angle.

→ Azimuthal symmetry is present, and no field components may vary with ϕ.

→ If we maintain ρ and ϕ constant while moving up and down the line charge by changing z, the line charge recedes into infinite distance in both directions and the problem is unchanged. This is axial symmetry and leads to fields which are not functions of z.

→ If we maintain ϕ and z constant and vary ρ, the problem changes and Coulomb's law leads us to expect the field to become weaker as ρ increases.

*→ No element of charge produces a ϕ component of Electric field Intensity: E_ϕ is zero. However, each element does produce an E_ρ and E_z component, but the contribution to E_z by elements of charge which are equal distances above and below the point at which we are determining the field will cancel.

→ Therefore, we have only an E_ρ component and it varies only with ρ.

→ We choose a point $P(0,y,0)$ on the y-axis at which to determine the field. This is a perfectly general point in view of the lack of variation of the field with ϕ and z.

To find the incremental field at P due to the incremental charge $dQ = \rho_L dz'$, we have

$$dE = \frac{\rho_L dz' \, (r-r')}{4\pi\epsilon_0 |r-r'|^3}$$

where $r = y a_y = \rho a_\rho$ & $r' = z' a_z$

so $r - r' = \rho a_\rho - z' a_z$ & $|r-r'| = \sqrt{\rho^2 + z'^2}$

Therefore,

$$dE = \frac{\rho_L dz' \, (\rho a_\rho - z' a_z)}{4\pi\epsilon_0 (\rho^2 + z'^2)^{3/2}}$$

Since only the E_ρ component is present, we may simplify:

$$dE_\rho = \frac{\rho_L \rho\, dz'}{4\pi\epsilon_0 (\rho^2+z'^2)^{3/2}}$$

→ Let $z' = \rho \cot\theta$ [from the figure]

$\Rightarrow dz' = -\rho \operatorname{cosec}^2\theta\, d\theta$

from the figure, $R = \rho \operatorname{cosec}\theta$.

→ To find the total Electric field at a particular point, we have to integrate the field due to individual Element over the entire length of line.

$$E_\rho = \int_{-\infty}^{\infty} \frac{\rho_L dz' \cdot \rho}{4\pi\epsilon_0 (\rho^2+z'^2)^{3/2}}$$

Since we assumed, $z' = \rho\cot\theta$

at $z' = -\infty \Rightarrow -\infty = \rho\cot\theta \Rightarrow \theta = \pi$

$z = \infty \Rightarrow \infty = \rho\cot\theta \Rightarrow \theta = 0$.

$$\therefore E_\rho = \frac{\rho_L \cdot \rho}{4\pi\epsilon_0}\int_{\pi}^{0} \frac{-\rho \operatorname{cosec}^2\theta\, d\theta}{[\rho^2+\rho^2\cot^2\theta]^{3/2}} = \frac{-\rho_L \rho^2}{4\pi\epsilon_0 \cdot \rho^3}\int_{+\pi}^{0}\frac{\operatorname{cosec}^2\theta\, d\theta}{[1+\cot^2\theta]^{3/2}}$$

$$= \frac{-\rho_L}{4\pi\epsilon_0\rho}\int_{+\pi}^{0}\frac{\operatorname{cosec}^2\theta}{\operatorname{cosec}^3\theta}\,d\theta = \frac{-\rho_L}{4\pi\epsilon_0\rho}\int_{+\pi}^{0}\frac{1}{\operatorname{cosec}\theta}\,d\theta$$

$$= \frac{-\rho_L}{4\pi\epsilon_0\rho}\int_{+\pi}^{0}\sin\theta\, d\theta = \frac{-\rho_L}{4\pi\epsilon_0\rho}\left[-\cos\theta\right]_{\pi}^{0}$$

$$= \frac{\rho_L}{4\pi\epsilon_0\rho}\left[\cos 0 - \cos\pi\right] = \frac{\rho_L}{4\pi\epsilon_0\rho}\left[1-(-1)\right] = \frac{\rho_L\,(2)}{4\pi\epsilon_0\rho}$$

**

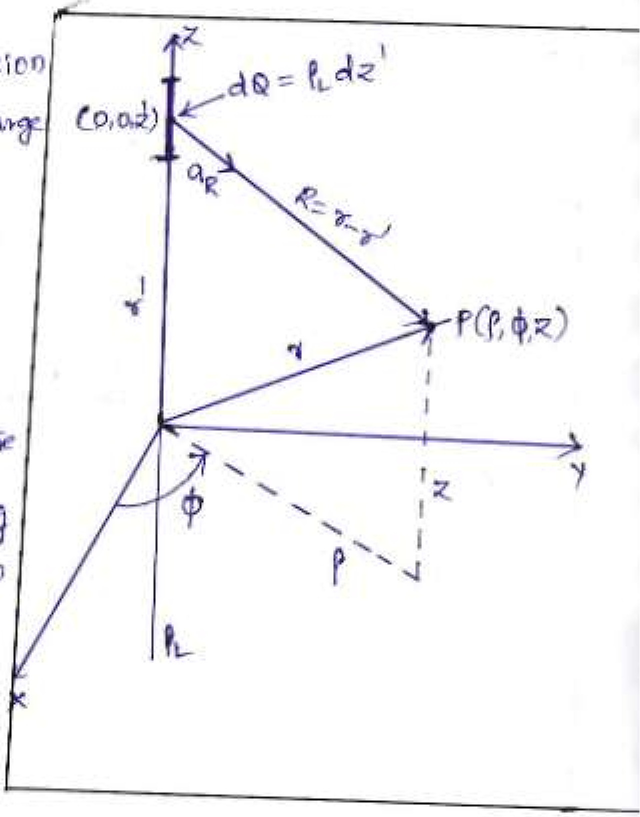
$$\therefore \boxed{E_\rho = \frac{\rho_L}{2\pi\epsilon_0\rho}}$$

↳ distance between origin to desired point of field

* This is the expression due to line charge for Electric field.

** Instead of taking cos) finding the Electric field on the Y-axis, if we consider a general point $P(\rho, \phi, z)$, at this point we have to find the total Electric field due to the length of line charge.

→ We might also have considered an expression for the Electric field due to volume charge as our starting point.



$$E = \int_{vol} \frac{\rho_v dv' (r - r')}{4\pi\epsilon_0 |r - r'|^3}$$

Let $\rho_v dv' = \rho_L dz'$ and integrating along the line which is now our volume containing all the charge. we assumed a point p now at a general location (ρ, ϕ, z) and write

$$r = \rho a_\rho + z a_z$$

$$r' = z' a_z$$

$$R = r - r' = \rho a_\rho + (z - z') a_z$$

$$|R| = \sqrt{\rho^2 + (z - z')^2} \qquad a_R = \frac{R}{|R|} = \frac{\rho a_\rho + (z - z') a_z}{\sqrt{\rho^2 + (z - z')^2}}$$

∴ Total field Expression E is

$$E = \int_{-\infty}^{\infty} \frac{\rho_L dz' \left[\rho a_\rho + (z - z') a_z\right]}{4\pi\epsilon_0 \left[\rho^2 + (z - z')^2\right]^{3/2}}$$

$$= \int_{-\infty}^{\infty} \frac{\rho_L \cdot \rho dz' a_\rho}{4\pi\epsilon_0 \left[\rho^2 + (z - z')^2\right]^{3/2}} + \int_{-\infty}^{\infty} \frac{\rho_L \cdot (z - z') dz' \; a_z}{4\pi\epsilon_0 \left[\rho^2 + (z - z')^2\right]^{3/2}}$$

$$= \frac{\rho_L}{4\pi\epsilon_0} \left[a_\rho \int_{-\infty}^{\infty} \frac{\rho dz'}{\left[\rho^2 + (z - z')^2\right]^{3/2}} + a_z \int_{-\infty}^{\infty} \frac{(z - z') dz'}{\left[\rho^2 + (z - z')^2\right]^{3/2}} \right]$$

Before integrating a vector expression, we must know whether or not a vector under the integral sign (here the unit vectors a_ρ and a_z) varies with the variable of integration (here dz'). If it does not, then it is a constant and may be removed from within the integral, leaving a scalar which may be integrated by normal methods. The direction of a_ρ doesnot change with z' (nor with ρ, but it does change with ϕ) and a_z is constant always. Hence we remove the unit vectors from the integrals and again integrate with tables or by changing variables,

$$E = \frac{\rho_L}{4\pi\epsilon_0}\left\{a_\rho \int_{-\infty}^{\infty} \frac{\rho\, dz'}{[\rho^2+(z-z')^2]^{3/2}} + a_z \int_{-\infty}^{\infty} \frac{(z-z')\,dz'}{[\rho^2+(z-z')^2]^{3/2}}\right\}$$

$$= \frac{\rho_L}{4\pi\epsilon_0}\left\{\left[a_\rho \cdot \rho \cdot \frac{1}{\rho^2}\frac{-(z-z')}{\sqrt{\rho^2+(z-z')^2}}\right]_{-\infty}^{\infty} + \left[a_z \frac{1}{\sqrt{\rho^2+(z-z')^2}}\right]_{-\infty}^{\infty}\right\}$$

$$= \frac{\rho_L}{4\pi\epsilon_0}\left[a_\rho \cdot \frac{2}{\rho} + a_z(0)\right]$$

$$= \frac{\rho_L}{2\pi\epsilon_0 \rho} a_\rho .$$

→ The resultant is same irrespective of different methods (or) assumptions.

∴ If a charge is distributed with uniform density ρ_L (C/m) along an infinite, straight line – which will be chosen as the z-axis – then the field is given by

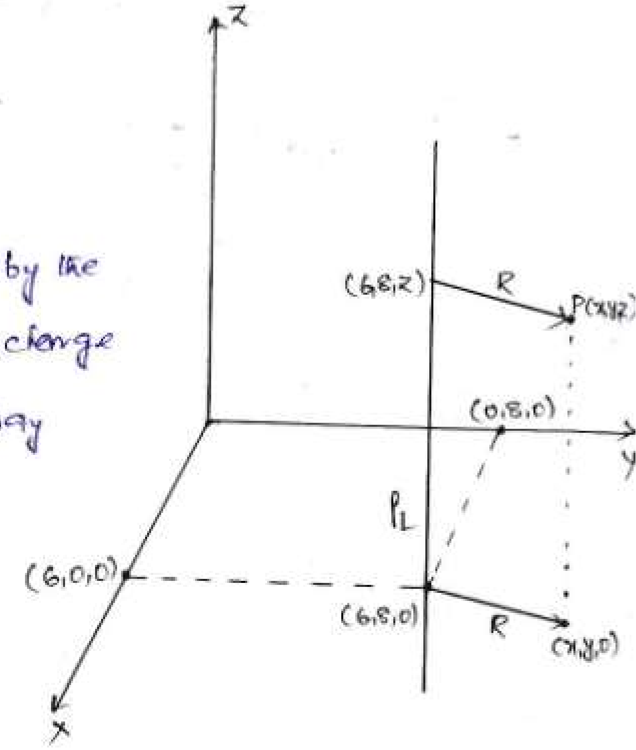
***

$$\boxed{E = \frac{\rho_L}{2\pi\epsilon_0 \rho} a_\rho}$$ (Cylindrical coordinates)

→ This field is cylindrical symmetry and is inversely proportional to the first power of the distance from the line charge.

→ As compared with the point charge, where the field decreased with the square of the distance. Moving ten times as far from a point charge leads to a field only 1 percent the previous strength, but moving ten times as far from a line charge only reduces the field to 10 percent of it's former value.

Prob:- Consider an infinite line charge parallel to the z-axis at $x=6, y=8$ shown in figure. find the $\bar{E}$ at the general field point $P(x,y,z)$?



Sol:- $$\bar{E} = \frac{\rho_L}{2\pi\epsilon_0 \rho} \cdot a_\rho$$

we replace ρ in the above eqn by the radial distance between the line charge and point P. $\rho = (x-6)a_x + (y-8)a_y$

$$\therefore |\rho| = \sqrt{(x-6)^2+(y-8)^2}$$

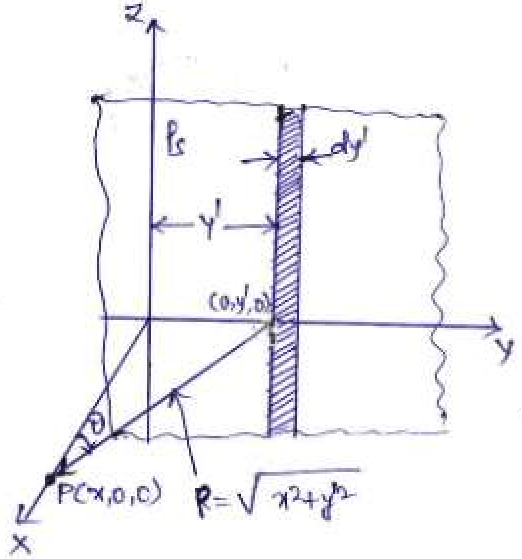
$$\bar{E} = \frac{\rho_L}{2\pi\epsilon_0 \sqrt{(x-6)^2+(y-8)^2}} a_\rho$$

where $$a_\rho = \frac{\rho}{|\rho|} = \frac{(x-6)a_x+(y-8)a_y}{\sqrt{(x-6)^2+(y-8)^2}}$$

$$\therefore \boxed{\bar{E} = \frac{\rho_L}{2\pi\epsilon_0} \frac{(x-6)a_x+(y-8)a_y}{(x-6)^2+(y-8)^2}}$$

Here the field is not a function of 'z'.

Field due to sheet of charge :—



→ Let us assume an infinite sheet of charge having a uniform density of ρ_s C/m² is placed in the yz plane and consider the symmetry. So that the field cannot vary with y or with z, and then that the y and z components arising from differential elements of charge symmetrically located with respect to the point at which we desired to find the field will cancel. Hence only E_x is present, and this component is a function of x alone.

→ Let us use the field of the infinite line charge since this infinite sheet of charge can be devided into infinite line charges i.e. by dividing the infinite sheet into differential width strips. One such strip is shown in the figure above.

The line charge density or charge per unit length is $\rho_L = \rho_s\, dy'$ and the distance from this line charge to the general point 'p' [at which we desired to find the field] on the x-axis is $R = \sqrt{x^2+y'^2}$.

→ The contribution to E_x at p from this differential-width strip is then

$$dE_x = \frac{\rho_s\, dy'}{2\pi\epsilon_0\sqrt{x^2+y'^2}}\cos\theta$$

from the figure, $\cos\theta = \frac{x}{\rho} = \frac{x}{\sqrt{x^2+y'^2}}$

$$\therefore \boxed{dE_x = \frac{\rho_s \cdot x\, dy'}{2\pi\epsilon_0(x^2+y'^2)}}$$

This is due to the single strip of line charge.

$\because dE = \frac{\rho_L}{2\pi\epsilon_0|\rho|}\cdot\frac{\vec{\rho}}{|\vec{\rho}|} = \frac{\rho_L}{2\pi\epsilon_0\rho^2}\cdot\vec{\rho}$

$dE = \frac{\rho_L}{2\pi\epsilon_0}\frac{[x a_x - y' a_y]}{(\sqrt{x^2+y'^2})^2}$

∵ field will cancel along y because of symmetry.

$dE = \frac{\rho_s dy'}{2\pi\epsilon_0}\cdot\frac{x a_x}{(x^2+y'^2)}$ ∵ $\rho_L = \rho_s dy'$

$E = \int_{-\infty}^{\infty}\frac{\rho_s x\, dy'\, a_x}{2\pi\epsilon_0(x^2+y'^2)}$

$E = \frac{\rho_s}{2\pi\epsilon_0} a_x \int_{-\infty}^{\infty}\frac{x\, dy'}{(x^2+y'^2)}$

→ Total field due to all the strip of line charges is

$$E_x = \frac{\rho_s}{2\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{x\,dy'}{x^2+y'^2}$$

$$= \frac{\rho_s}{2\pi\epsilon_0}\left[\tan^{-1}(y'/x)\right]_{-\infty}^{\infty} = \frac{\rho_s}{2\pi\epsilon_0}\left[\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right]$$

$$= \frac{\rho_s}{2\pi\epsilon_0}\left[\pi/2 + (\pi/2)\right] = \frac{\rho_s}{2\pi\epsilon_0}[\pi]$$

$$\left\{ \tan^{-1}(x/a) = \frac{1}{a}\cdot\frac{1}{1+(x/a)^2} = \frac{1}{a}\cdot\frac{a^2}{x^2+a^2} = \frac{a}{x^2+a^2} \right.$$

$$E_x = \frac{\rho_s \cdot \pi}{2\pi\epsilon_0}$$

$$\boxed{E_x = \frac{\rho_s}{2\epsilon_0}} \quad ***$$

This is the expression for total field due to the sheet of charge on the positive x-axis.

→ If the point 'p' were chosen on the negative x-axis, then

$$*** \quad \boxed{E_x = \frac{-\rho_s}{2\epsilon_0}}.$$

for the field is always directed away from the positive charge. This difficulty in sign is usually overcome by specifying a unit vector a_N, which is normal to the sheet and directed outward, or away from it. Then

$$\boxed{E = \frac{\rho_s}{2\epsilon_0}\,a_N} \quad ***$$

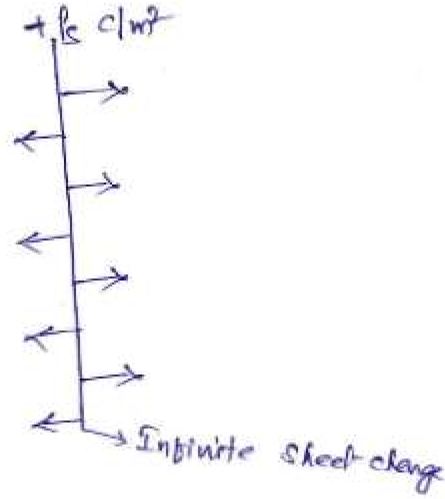
→ This field is constant in Magnitude and direction.

→ Conclusions :-

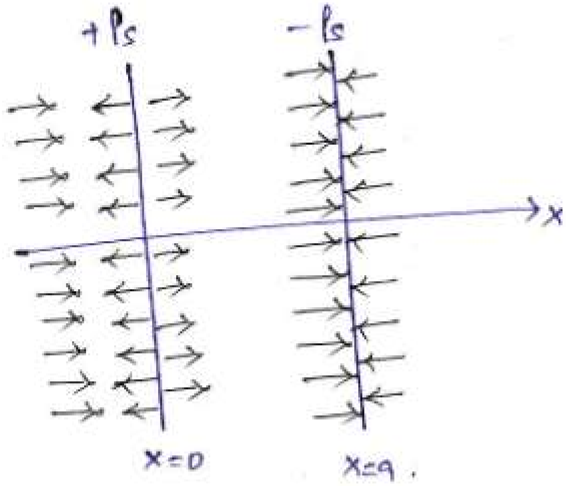
* E-field lines are always normal to the sheet and parallel to themselves.

* E- is uniform field for a sheet charge.

* Since half of the field lines are going to left and half are going to right. so surface charge density is distributed on either side of the sheet so we put '2' in the denominator.



* Assume a second infinite sheet of charge, having a negative charge density $-\rho_s$ is located in the plane $x=a$. To find the total field we have to add the contribution of each sheet.



→ In the region $x>a$:

$$E_+ = \frac{\rho_s}{2\epsilon_0} a_x$$

$$E_- = \frac{-\rho_s}{2\epsilon_0} a_x$$

$$\therefore E = E_+ + E_- = \frac{\rho_s a_x}{2\epsilon_0} - \frac{\rho_s}{2\epsilon_0} a_x = 0.$$

$$\therefore E = 0 \quad [x>a]$$

→ In the region $x<0$:

$$E_+ = \frac{-\rho_s}{2\epsilon_0} a_x \quad , \quad E_- = \frac{+\rho_s}{2\epsilon_0} a_x.$$

$$\therefore E = E_+ + E_- = \frac{-\rho_s}{2\epsilon_0} a_x + \frac{+\rho_s}{2\epsilon_0} a_x = 0.$$

$$\therefore E = 0 \quad [x<a]$$

→ In the region $0<x<a$:

$$E_+ = \frac{\rho_s}{2\epsilon_0} a_x \quad , \quad E_- = \frac{\rho_s}{2\epsilon_0} a_x$$

$$\therefore E = E_+ + E_-$$

$\therefore E = E_+ + E_-$

$= \frac{\rho_s}{2\epsilon_0} a_x + \frac{\rho_s}{2\epsilon_0} a_x = \frac{\rho_s}{\epsilon_0} a_x.$

$\therefore \boxed{E = \frac{\rho_s}{\epsilon_0} a_x}$

This is an expsession for the field between the parallel plates of an air capacitor, provided the linear dimensions of the plates are very much greater than their separation

** The field out-side the capacitor is not zero, but it is negligible.

Prob:— An infinitely long, uniform line charge is located at $y=3, z=5$. If $\rho_L = 30\,nC/m$ find E at: (a) The origin (b) $P_B(0,6,1)$ (c) $P_C(5,6,1)$ → ?

Sol:— $\rho_L = 30\,nC/m$, $O(0,3,5)$

General formula is $E = \frac{\rho_L}{2\pi\epsilon_0|\rho|} a_\rho$ where $a_\rho = \frac{\rho}{|\rho|}$

(a) $P_A(0,0,0)$ $O(0,3,5)$

$\rho = (0-0)a_x + (0-3)a_y + (0-5)a_z$

$= -3a_y - 5a_z$

$|\rho| = \sqrt{9+25} = \sqrt{34}$

$E = \frac{30 \times 10^{-9}}{2\pi \times \frac{10^{-9}}{36\pi} \times \sqrt{34}} \cdot \frac{(-3a_y - 5a_z)}{\sqrt{34}}$

$= \frac{30 \times 18}{34} [-3a_y - 5a_z]$

$= 15.88 [-3a_y - 5a_z]$

$= -47.6a_y - 79.3a_z$ V/m

(b) $O(0,3,5)$ $P_B(0,6,1)$

$\rho = 3a_y - 4a_z$

$|\rho| = \sqrt{9+16} = 5$

$E = \frac{30 \times 10^{-9}}{2\pi \times \frac{10^{-9}}{36\pi} \times 5} \frac{[3a_y - 4a_z]}{5}$

$= \frac{18 \times 30}{25} [3a_y - 4a_z] = 21.6 [3a_y - 4a_z]$

$= 64.8a_y - 86.4a_z$ V/m

(c) $O(0,3,5)$ $P_C(5,6,1)$

$\rho = 5a_x + 3a_y - 4a_z$, $|\rho| = \sqrt{25+9+16} = \sqrt{50}$

$E = \frac{18 \times 30}{\sqrt{50}} \frac{[5a_x + 3a_y - 4a_z]}{\sqrt{50}}$

$= \frac{18 \times 30}{50} [5a_x + 3a_y - 4a_z]$

$= 54a_x + 32.4a_y - 43.2a_z$ V/m.

Electric flux :— (denoted by ψ).

→ We know that at any particular point the Electric field strength $\vec{E}$ depends not only upon the magnitude and position of the charge 'Q', but also upon the dielectric constant of the medium (air, oil and others) in which the field is measured. It is desirable to associate with the charge 'Q' a second Electrical Quantity that will be independent of the medium involved. This second Quantity is called 'Electric Displacement (or) Electric flux'.

→ An understanding of concept of Electric Displacement by faraday's Experiments with concentric spheres. According to faraday, A sphere with charge 'Q' was placed within, but not touching, a larger hollow sphere. The outer sphere was earthed on momentarily and then the inner sphere was removed. The charge remaining on the outer sphere was measured. This charge was found to be equal (and of opposite sign) to the charge on the inner sphere for all sizes of the spheres and for all types of dielectric media between the spheres. Thus it could be considered that there was an Electric Displacement from the charge on the inner sphere through the medium to the outer sphere, the amount of this displacement depending only upon the magnitude of the charge 'Q'. The displacement 'ψ' is equal in magnitude to the charge that produces it, that is

** $$\boxed{\psi = Q}$$

→ for the case of an isolated point charge 'Q' remote from other bodies the outer sphere is assumed to have infinite radius.

Electric flux Density :— [D]

→ Electric flux Density can also called as "Electric Displacement Density". Denoted by 'D'.

→ Electric flux Density is defined as 'Electric Displacement per unit Area' (or) 'flux per unit Area'.

→ Electric flux Density (D) at any point on a spherical surface of radius 'R' centered at the isolated charge 'Q' will be

$$\boxed{D = \frac{\psi}{4\pi R^2} = \frac{Q}{4\pi R^2}}$$ coulomb/sqr mts

→ since $D = \frac{\psi}{Area} = \frac{Q}{Area}$

if the Radius of the sphere is 'R', then the Area of the sphere is $= 4\pi R^2$

$$\therefore D = \frac{Q}{4\pi R^2}$$

→ The Displacement per unit area at any point depends upon the direction of the Area. So 'D' is therefore a vector Quantity, its direction being taken as that direction of the normal to the surface element which makes the displacement through the element of area maximum.

→ for the case of displacement from an isolated charge this direction is along the radial from the charge and is the same as the direction of 'E'. Therefore

$$D = \frac{Q}{4\pi R^2} a_R$$

*> The direction of 'D' at a point is the direction of the flux lines at that point, and the magnitude is given by the number of flux lines crossing a surface normal to the lines divided by the surface Area.

→ The number of lines per unit Area is made proportional to the magnitude of the Electric field strength, $\bar{E}$.

→ A line of electric flux is a curve drawn so that at every point it has the direction of the Electric flux density (or) Displacement Density.

→ The no. of flux lines per unit Area is used to indicate the magnitude of the displacement Density.

→ In Homogeneous isotropic media, lines of force and lines of flux always have the same direction.

→ If the radius of sphere is R=a, then any point on the surface of the sphere Electric flux Density D is

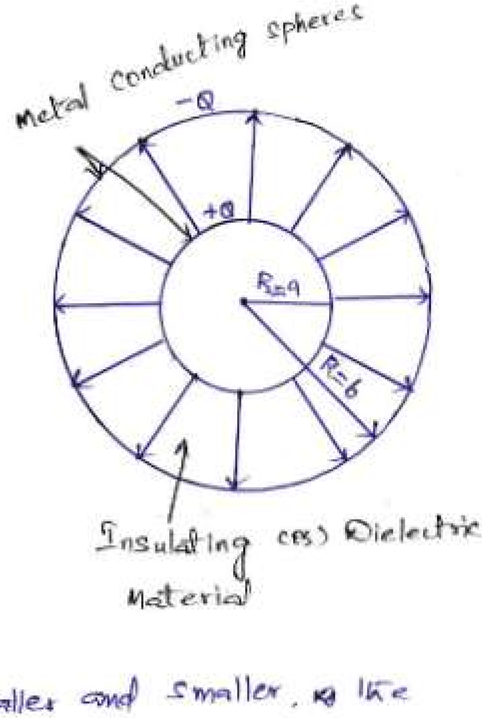
$$D\Big|_{r=a} = \frac{Q}{4\pi a^2} a_r \text{ (inner sphere)}$$

If the radius of sphere is R=b, then D is

$$D\Big|_{r=b} = \frac{Q}{4\pi b^2} a_r \text{ (outer sphere)}$$

and at a radial distance r, where $a\le r\le b$.

$$D = \frac{Q}{4\pi R^2} a_r.$$



→ If we now let the inner sphere becomes smaller and smaller, as the inner sphere has the charge Q, It becomes a point charge in the limit. But the Electric flux Density at a point R metres from the point charge is given by

$$\boxed{D = \frac{Q}{4\pi R^2} a_r} \longrightarrow (1)$$

for Q lines of flux are symmetrically directed outward from the point and pass through an imaginary spherical surface of area $4\pi R^2$.

→ since we know that the Electric field intensity at a distance 'R' from the point charge is (in free space)

$$\boxed{E = \frac{Q}{4\pi R^2 \cdot \epsilon_0} \cdot a_R} \longrightarrow (2)$$

where $\epsilon_0 \rightarrow$ free space permittivity

By comparing eqns (1) & (2),

$$E = \frac{D}{\epsilon_0} \Rightarrow \boxed{D = \epsilon_0 E} \quad \text{(for free space only)} \longrightarrow (3)$$

Expression for 'D' is applicable to vacuum only.

→ for a general volume charge distribution in free space is

$$\boxed{E = \int_{vol} \frac{\rho_v dv}{4\pi\epsilon_0 R^2} a_R} \longrightarrow (4) \quad \text{(free space)}$$

from eqn (3) & (4)

$$\boxed{D = \int_{vol} \frac{\rho_v dv}{4\pi R^2} a_R} \longrightarrow (5)$$

→ faradays results shows that eqn (1) is still applicable, and thus so is eqn (5). Eqn (4) is not applicable, however, and so the relationship between 'D and E' will be slightly more complicated than eqn (3).

Prob in Hayt

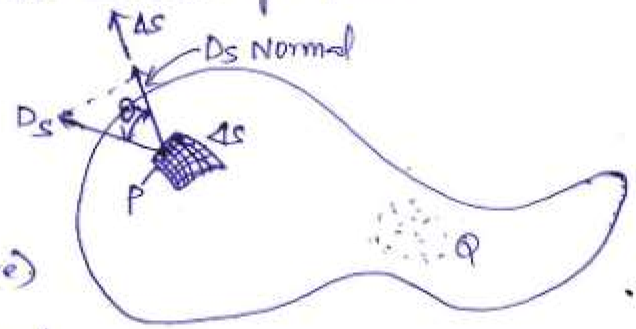
EMTL U-I-(17)

Gauss's Law:-

Gauss's Law states that "the Electric flux passing through an closed imaginary spherical surface lay, is equal to the total charge enclosed by that surface".

→ According to Gauss's, the electric flux passing through any imaginary spherical surface lying between the two conducting spheres is equal to the charge enclosed with in that imaginary surface. This enclosed charge is distributed on the surface of the inner sphere, or it might be concentrated as a point charge at the center of the imaginary sphere. However, since one coulomb of electric flux is produced by one coulomb of charge.

→ +Q coulombs on any inner conductor would produce an induced charge of −Q coulombs on the surrounding sphere.



→ Let us imagine a distribution of charge, shown as a cloud of point charges (as shown in figure) surrounded by a closed surface of any shape. The closed surface may be the surface of some real material. If the total charge is Q, then Q coulomb's of electric flux will pass through the enclosing surface. At every point on the surface the electric flux density D vector will have some value D_S, where the subscript 's' merely reminds us that D must be evaluated at the surface and D_S will in general vary in magnitude and direction from one point on the surface to another.

→ Consider the nature of an incremental element of the surface. An incremental element of area ΔS is very nearly a portion of a plane surface, and the complete description of this surface element requires not only a statement of it's magnitude as but also of it's orientation in space. The incremental surface element is a vector Quantity. The only unique direction which may be associated with Δs is the direction of the normal to that plane which is tangent to the surface at the point.

→ At any point 'P' consider an incremental element of surface Δs and let D_s make an angle 'θ' with Δs as shown in figure. The flux crossing Δs is then the product of the normal component of D_s and Δs.

$$\Delta \psi = \text{flux crossing } \Delta s = D_{s,norm} \Delta s = D_s \cos\theta \, \Delta s = D_s \cdot \Delta s$$

→ The total flux passing through the closed surface is obtained by adding the differential contributions crossing each surface element Δs,

$$\boxed{\psi = \int d\psi = \oint_{\substack{closed\\surface}} D_s \cdot ds}$$

The resultant integral is a closed surface integral, and since the surface element "ds" always involves the differentials of two coordinates, such as $dx\,dy$, $\rho\, d\phi\, d\rho$, (or) $r^2 \sin\theta\, d\theta\, d\phi$, the integral is a double integral.

→ The purpose of place a small circle on the integral sign itself to indicate that the integration is to be performed over a closed surface. Such a surface is often called a 'Gaussian surface'. Then we have the mathematical formulation of Gauss's Law,

$$\boxed{\psi = \oint_s D_s \cdot ds = \text{charge enclosed} = Q}$$

→ The charge enclosed might be several point charges, in such case

$$Q = \sum Q_n$$

or a line charge, $Q = \int \rho_L \cdot dL$

(ii) a Surface charge,

$$\boxed{Q = \int_S \rho_s \cdot ds}$$ (not necessarily a closed surface)

(iii) a volume charge distribution,

$$\boxed{Q = \int_{vol} \rho_v \cdot dv}$$

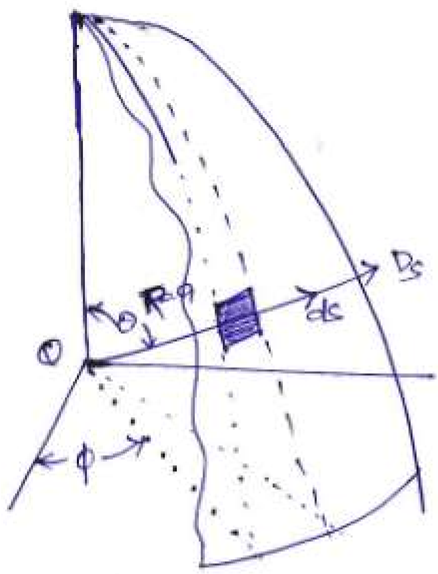
→ Gauss's Law may be written interms of the charge distribution as

$$\boxed{\oint_S D_s \cdot ds = \int_{vol} \rho_v \cdot dv}$$

→ The mathematical statement means that the total electric flux through any closed surface is equal to the charge enclosed.

Application of Gauss's Law:—

(i) To the field of a point charge:—



→ To illustrate the application, place a point charge 'Q' at the origin of a spherical coordinate system and by choosing our closed surface as a sphere of radius 'a'.

→ The electric field intensity of the point charge at a distance R from the point charge Q which is at the origin.

$$E = \frac{Q}{4\pi\epsilon_0 R^2} a_R$$

and since $D = \epsilon_0 E$

$$\therefore \quad D = \frac{Q}{4\pi R^2} a_R .$$

At the surface of the sphere.

$$D_s = \frac{Q}{4\pi a^2} a_R$$

→ The differential Element of area on a spherical Surface is, in spherical coordinates,

$$ds = r^2 \sin\theta \, d\theta \, d\phi = a^2 \sin\theta \, d\theta \, d\phi$$

$$\therefore ds = a^2 \sin\theta \, d\theta \, d\phi \, a_r$$

The Integrand is

$$D_s \cdot ds = \frac{Q}{4\pi a^2} \cdot a^2 \sin\theta \, d\theta \, d\phi \, a_r \cdot a_r = \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

Leading to the closed surface integral.

$$\oint_{sph} D_s \cdot ds = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi = \frac{Q}{4\pi} \int_{\phi=0}^{2\pi} \left[-\cos\theta\right]_0^{\pi} d\phi$$

$$= \frac{Q}{4\pi} \left[-\cos\pi + \cos 0\right] \int_{\phi=0}^{2\pi} d\phi = \frac{Q}{4\pi} [2] \left[\phi\right]_0^{2\pi}$$

$$= \frac{Q}{4\pi} [2][2\pi] = \frac{Q}{4\pi} [4\pi]$$

$$\boxed{\int D_s \cdot ds = Q}$$

It shows that Q coulombs of electric flux are crossing the surface, since the enclosed charge is Q coulombs.

→ The solution of Gauss's Law is easy if we are able to choose a closed surface which satisfies two conditions:

(1) D_s is everywhere either normal or tangential to the closed surface, so that $D_s \cdot ds$ becomes either $D_s ds$ or zero, respectively.

(2) on the portion of the closed surface for which $D_s \cdot ds$ is not zero, D_s = constant.

This allows us to replace the dot product with the product of the scalars D_s and ds and then to bring D_s outside the integral sign. The remaining integral is then $\int_s ds$ over that portion of the closed

Surface which D_s crosses normally, and this is simply the area of this section of that surface.

→ The electric field intensity due to a positive point charge is directed radially outward from the point charge.

→ Let us again consider a point charge Q at the origin of a spherical coordinate system and decide on a suitable closed surface which will meet the two requirements. D_s is everywhere normal to the surface; D_s has the same value at all points on the surface. Then we have,

$$Q = \oint_S D_s \cdot ds = \oint_{sph} D_s ds = D_s \oint_{sph} ds = D_s \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$Q = 4\pi r^2 D_s$$

and hence $\boxed{D_s = \frac{Q}{4\pi r^2}}$

Since r may have any value and since D_s is directed radially outward,

$$\boxed{D = \frac{Q}{4\pi r^2} a_r \qquad E = \frac{Q}{4\pi \epsilon_0 r^2} a_r.}$$

(ii) To the field of a line charge:—

→ Let us consider the uniform line charge distribution ρ_L lying along the z-axis and extending from $-\infty$ to $+\infty$. We must first obtain a knowledge of the symmetry of the field.

→ Application of Gauss's Law depends on symmetry. If we cannot show that symmetry exists then we cannot use Gauss's Law to obtain a solution.

→ Since for the uniform line charge, it is evident that only the radial component of D is present or

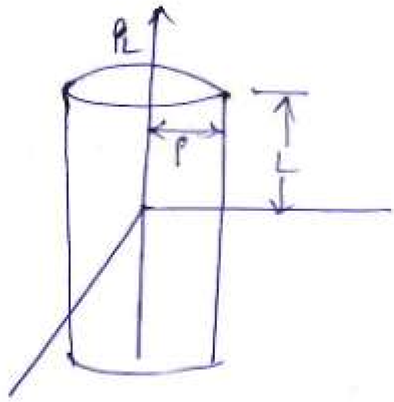
$$D = D_\rho a_\rho$$

and this component is a function of ρ only.

$$D_\rho = f(\rho)$$

The choice of a closed surface for a cylindrical surface is the only surface to which D_ρ is everywhere normal and it may be closed by plane surfaces normal to the z-axis. A closed right circular cylindrical of radius ρ extending from $z=0$ to $z=L$ is

we apply Gauss's Law,



$$Q = \oint_{cyl} D_s \cdot ds = D_s \int_{sides} ds + 0 \int_{top} ds + 0 \int_{bottom} ds$$

$$= D_s \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho \, d\phi \, dz = D_s \cdot 2\pi \rho L$$

and obtain

$$\boxed{D_s = D_\rho = \frac{Q}{2\pi \rho L}}$$

Interms of the charge density ρ_L, the total charge enclosed is

$$Q = \rho_L L$$

$$\boxed{D_\rho = \frac{\rho_L}{2\pi \rho}}$$

Since $D = \epsilon_0 E$

$$\therefore \boxed{E_\rho = \frac{\rho_L}{2\pi \epsilon_0 \rho}}$$

So Gauss's Law has been proved for line charge.

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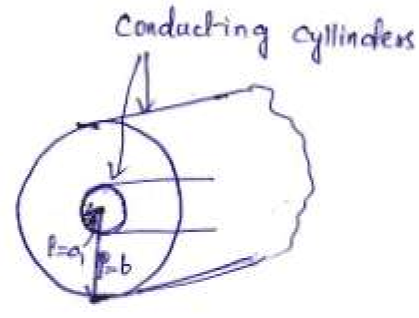
(iii) for Surface charge:-

Suppose that if we have two coaxial cylindrical conductors, the inner of radius 'a' and the outer of radius 'b', each infinite in extent from the fig.



we shall assume a charge distribution of ρ_s on the outer surface of the inner conductor.

→ Symmetry considerations show us that only the D_ρ component is present and that it can be a function only of ρ. A right circular cylinder of length 'L' and radius 'ρ', where $a<\rho<b$, is necessarily chosen as the gaussian surface and we have

$$Q = D_s \cdot 2\pi\rho L$$

The total charge on a length 'L' of the inner conductor is

$$Q = \int_{z=0}^{L}\int_{\phi=0}^{2\pi} \rho_s \cdot a\, d\phi\, dz = 2\pi a L \rho_s$$

from which we have

$$D_s = \frac{a\rho_s}{\rho} \quad , \quad \boxed{D = \frac{a\rho_s}{\rho} a_\rho} \ (a<\rho<b)$$

This result might be expressed interms of charge per unit length, because the inner conductor has $2\pi a \rho_s$ coulombs on a meter length, and hence letting $\rho_L = 2\pi a \rho_s$,

$$\therefore \boxed{D = \frac{\rho_L}{2\pi\rho} \cdot a_\rho}$$

The solution has a form identical with that of the infinite line charge.

→ Since Everyline of electric flux starting from the charge on the inner cylinder must terminate on a negative charge on the inner surface of the outer cylinder, the total charge on that surface must be

$$\boxed{Q_{outer\ cyl} = -2\pi a L \rho_s, \text{inner cyl}}$$

and the surface charge on the outer cylinder is found as

$$2\pi b L \rho_{s,\text{outer cyl}} = -2\pi a L \rho_{s,\text{inner cyl}}$$

(or) $$\boxed{\rho_{s,\text{outer cyl}} = -\frac{a}{b}\rho_{s,\text{inner cyl}}}$$

→ If we use a cylinder of radius ρ & $\rho > b$ for the Gaussian surface, the total charge enclosed is zero, for there are equal and opposite charges on each conducting cylinder. Hence

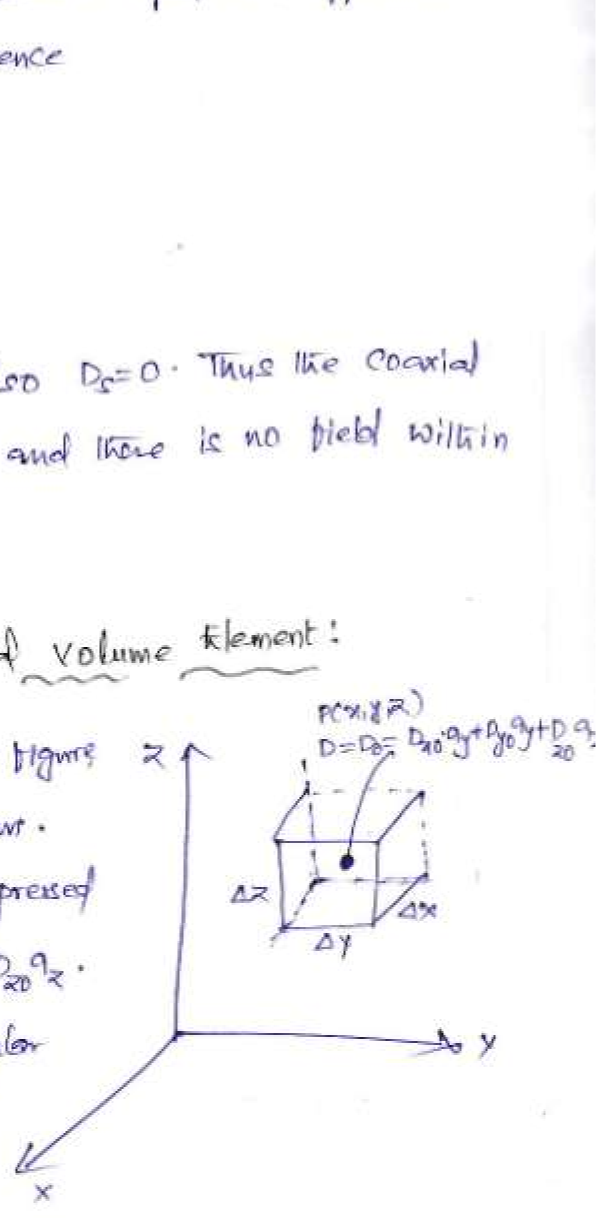
$$0 = D_s 2\pi\rho L \qquad (\rho > b)$$

$$D_s = 0 \qquad (\rho > b)$$

→ If we take $\rho < a$, in such case also $D_s = 0$. Thus the coaxial cable or capacitor has no external field and there is no field within the center conductor.

(iv) Application of Gauss's Law for Differential volume Element:

→ Let us consider any point P, shown in figure located by a cartesian coordinate system. The value of $\vec{D}$ at the point P may be expressed in cartesian components, $D_0 = D_{x0} a_x + D_{y0} a_y + D_{z0} a_z$. Here we choose a closed surface small rectangular box, centered at P, having sides of lengths Δx, Δy and Δz and apply Gauss's Law,



$$\oint_S D \cdot ds = Q$$

In order to evaluate the integral over the closed surface, the integral must be broken up into six integrals, one over each face.

$$\oint_s D\cdot ds = \int_{front} + \int_{back} + \int_{left} + \int_{right} + \int_{top} + \int_{bottom}$$

since the surface element is very small, D is essentially constant (over this portion of the entire closed surface) and

$$\int_{front} = D_{front}\cdot \Delta S_{front}$$

$$= D_{front}\cdot \Delta y \Delta z\, a_x$$

$$= D_{x,front}\, \Delta y\, \Delta z$$

where we have only to approximate the value of D_x at this front face. The front face is at a distance of $\Delta x/2$ from P, and hence

$$D_{x,front} = D_{x0} + \frac{\Delta x}{2}\times \text{rate of change of } D_x \text{ with } x$$

$$= D_{x0} + \frac{\Delta x}{2}\cdot\frac{\partial D_x}{\partial x}$$

where D_{x0} is the value of D_x at P, and where a partial derivative must be used to express the rate of change of D_x with x, since D_x in general also varies with y and z. This expression could have been obtained more formally by using the constant term and the term involving the first derivative in the Taylor's series expansion for D_x.

we have now

$$\int_{front} = \left(D_{x0} + \frac{\Delta x}{2}\cdot\frac{\partial D_x}{\partial x}\right)\Delta y\,\Delta z$$

consider now the integral over the back surface,

$$\int_{back} = D_{back}\cdot \Delta S_{back}$$

$$= D_{back}\cdot(-\Delta y\,\Delta z\, a_x)$$

$$= -D_{x,back}\,\Delta y\,\Delta z.$$

$$\therefore D_{x,back} = D_{x0} - \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}$$

giving

$$\int_{back} = \left(-D_{x0} + \frac{\Delta x}{2}\cdot\frac{\partial D_x}{\partial x}\right)\Delta y\,\Delta z$$

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$$\oint_S D\cdot ds = \int_{front} + \int_{back} + \int_{left} + \int_{right} + \int_{top} + \int_{bottom}$$

Since the surface element is very small, D is essentially constant (over this portion of the entire closed surface) and

$$\int_{front} = D_{front} \cdot \Delta S_{front}$$

$$= D_{front} \cdot \Delta y \Delta z\, a_x$$

$$= D_{x,front}\, \Delta y\, \Delta z$$

where we have only to approximate the value of D_x at this front face. The front face is at a distance of $\Delta x/2$ from P, and hence

$$D_{x,front} = D_{x0} + \frac{\Delta x}{2} \times \text{rate of change of } D_x \text{ with } x$$

$$= D_{x0} + \frac{\Delta x}{2} \cdot \frac{\partial D_x}{\partial x}$$

where D_{x0} is the value of D_x at P, and where a partial derivative must be used to express the rate of change of D_x with x, since D_x in general also varies with y and z. This expression could have been obtained more formally by using the constant term and the term involving the first derivative in the Taylor's series expansion for D_x.

we have now

$$\int_{front} = \left(D_{x0} + \frac{\Delta x}{2} \cdot \frac{\partial D_x}{\partial x}\right) \Delta y\, \Delta z$$

consider now the integral over the back surface,

$$\int_{back} = D_{back} \cdot \Delta S_{back}$$

$$= D_{back} \cdot (-\Delta y\, \Delta z\, a_x)$$

$$= -D_{x,back}\, \Delta y \Delta z.$$

$$\therefore D_{x,back} = D_{x0} - \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$$

giving

$$\int_{back} = \left(-D_{x0} + \frac{\Delta x}{2} \cdot \frac{\partial D_x}{\partial x}\right) \Delta y\, \Delta z$$

If we combine these two integrals, we have

$$\int_{front} + \int_{back} = \frac{\partial D_x}{\partial x} \Delta x \Delta y \Delta z$$

By exactly the same process we find that

$$\int_{right} + \int_{left} = \frac{\partial D_y}{\partial x} \cdot \Delta x \Delta y \Delta z$$

and

$$\int_{top} + \int_{bottom} = \frac{\partial D_z}{\partial z} \cdot \Delta x \Delta y \Delta z$$

and these results may be collected to yield

$$\oint_S D \cdot ds = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x \Delta y \Delta z$$

(or) $$\boxed{\oint_S D \cdot ds = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \cdot \Delta v}$$

The expression is an approximation which becomes better as Δv becomes smaller. We have applied Gauss's Law to the closed surface surrounding the volume element Δv and have as a result the above expression stating that

$$\boxed{\text{charge enclosed in volume } \Delta v = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) * \text{Volume } \Delta v}$$

Divergence:

Since we know that

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) = \frac{\oint_S D \cdot ds}{\Delta v} = \frac{Q}{\Delta v} \longrightarrow (1)$$

(or) as a limit

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) = \lim_{\Delta v \to 0} \frac{\oint D \cdot ds}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v} \longrightarrow (2)$$

Where the approximation has been replaced by an equality. It is evident that last term in the above equation is the volume charge density ρ_v and hence that

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint D\cdot ds}{\Delta v}=\rho_v$$

This eqn can be written as separate eqn's.

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint D\cdot ds}{\Delta v} \longrightarrow (3)$$

and $$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\rho_v \longrightarrow (4)$$

Eqn (3) does not involve charge density, and the methods used on any vector $\bar{A}$ to find $\oint_s \bar{A}\cdot ds$ for a small closed surface, leading to

$$\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}=\lim_{\Delta v\to 0}\frac{\oint_s \bar{A}\cdot ds}{\Delta v} \longrightarrow (5)$$

where 'A' is any vector field.

The above equation is called the "Divergence".

→ The divergence of a vector $\bar{A}$ is defined as

$$\text{Divergence of } \bar{A}=\text{div}\bar{A}=\lim_{\Delta v\to 0}\frac{\oint_s \bar{A}\cdot ds}{\Delta v}$$

** The divergence of the vector flux density $\bar{A}$ is the out-flow of flux from a small closed surface per unit volume as the volume shrinks to zero.

→ A positive divergence for any vector quantity indicates a source of that vector quantity at that point.

→ A Negative divergence for any vector quantity indicates a sink of that vector quantity at that point.

Expressions for divergence are

$$\text{div } D = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \quad \text{(Cartesian)}$$

$$\text{div } D = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \quad \text{(Cyllindrical)}$$

$$\text{div } D = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi} \quad \text{(spherical)}$$

→ Divergence describes that how much flux is leaving a small volume on a per-unit volume, no direction is associated with it.

Electric potential :-

Suppose to move a point charge 'Q' from point 'A' to point 'B' in an electric field $\bar{E}$, from Coulomb's Law, the force on Q is $F = Q\bar{E}$. So that the work done in displacing the charge by dl is

$$\boxed{dW = -F \cdot dl = -Q\bar{E} \cdot dl} \longrightarrow (1)$$

→ The negative sign indicates that the work is being done by an external agent. Thus the total work done, or the potential energy required, in moving 'Q' from 'A' to 'B', is

$$\boxed{W = -Q\int_A^B E \cdot dl} \longrightarrow (2)$$

Dividing 'W' by 'Q' in the above eqn gives the potential energy per unit charge. This quantity is denoted by V_{AB}, is known as the potential difference between points 'A' and 'B'. Thus

$$\boxed{V_{AB} = \frac{W}{Q} = -\int_A^B E \cdot dl} \longrightarrow (3)$$

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Notes :

(1) In determining V_{AB}, A is the initial point while B is the final point

(2) If V_{AB} is negative, there is a loss in potential energy in moving Q from A to B, this implies that the work is being done by the field. However, if V_{AB} is positive there is a gain in potential energy in the movement; an external agent performs the work.

(3) V_{AB} is independent of the path taken.

(4) V_{AB} is measured in Joules per Coulomb, commonly referred to as Volts.

(5) potential is always a scalar function.

→ The Electric field intensity (E) due to a point charge Q located at the origin is

$$E = \frac{Q}{4\pi\epsilon_0 r^2} a_r$$

from eqn (3),

$$V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2} a_r \cdot dr\, a_r = -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0 r^2} dr$$

$$= -\left[\frac{Q}{4\pi\epsilon_0}\right] \int_{r_A}^{r_B} \frac{1}{r^2} dr = -\frac{Q}{4\pi\epsilon_0} \left[-1/r\right]_{r_A}^{r_B}$$

***

$$\therefore \boxed{V_{AB} = \frac{Q}{4\pi\epsilon_0 r}\Big]_{r_A}^{r_B}}$$

***

$$\therefore \boxed{V_{AB} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]} \longrightarrow (4)$$

***

(or) $\boxed{V_{AB} = V_B - V_A} \longrightarrow (5)$

where $V_B = \frac{Q}{4\pi\epsilon_0 r_B}$, $V_A = \frac{Q}{4\pi\epsilon_0 r_A}$

→ Where V_B and V_A are the potentials (or Absolute potentials) at B and A respectively. Thus the potential difference V_{AB} may be regarded as the potential at B with reference to A.

→ If the separation between two point charges at is less then the potential between those two point charges is large and if the separation between two point charges is Large then the potential between those two point charges is small.

→ Assume If point A is at infinity and is Consider as reference. in such case $r_A = \infty$ i.e therefore $V_A = 0$, then the potential at any point $(r_B \to r)$ due to a point charge Q located at the origin is

$$⑥ \leftarrow \boxed{V = \frac{Q}{4\pi\epsilon_0 r}}$$

→ This expression defines the potential at any point distant r from a point charge Q at the origin, the potential at infinite radius being taken as the zero reference.

→ The potential at any point is the potential difference between that point and a chosen point (or reference point) at which the potential is zero.

→ By assuming zero potential at infinity, the potential at a distance r from the point charge is the work done per unit charge by an external agent in transfering a test charge from infinity to that point. Thus

$$\boxed{V = -\int_{-\infty}^{r} E \cdot dl} = \frac{W}{Q} \longrightarrow ⑦$$

Since
Work = -force × displacement
$dw = \bar{F} \cdot d\bar{l} = -Q\bar{E} \cdot dl$
$W = -Q\int E \cdot dl$

→ If the point charge Q in eqn ⑥ is not located at the origin but at a point whose position vector is r', the potential $V(x, y, z)$ or simply $V(r)$ at r becomes

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$$\boxed{V(r) = \frac{Q}{4\pi\epsilon_0 |r - r'|}} \longrightarrow (8)$$

→ If there are 'n' point charges $Q_1, Q_2, \dots Q_n$ which are located at points with position vectors $r_1, r_2, \dots r_n$ respectively then the potential at 'r' is [from the superposition principle],

$$V(r) = \frac{Q_1}{4\pi\epsilon_0 |r - r_1|} + \frac{Q_2}{4\pi\epsilon_0 |r - r_2|} + \dots + \frac{Q_n}{4\pi\epsilon_0 |r - r_n|}$$

(or) $$\boxed{V(r) = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{Q_k}{|r - r_k|}}$$ (point charges) $\longrightarrow (9)$

→ For continuous charge distributions, we replace Q_k in eqn (9) with charge element $\rho_L dl$, $\rho_s ds'$ or $\rho_v dv$ and the summation becomes an integration, so the potential 'V' at 'r' becomes

$$V(r) = \frac{1}{4\pi\epsilon_0} \int_L \frac{\rho_L(r')\, dl'}{|r - r'|} \quad \text{(line charge)}$$

$$V(r) = \frac{1}{4\pi\epsilon_0} \int_S \frac{\rho_S(r')\, ds'}{|r - r'|} \quad \text{(Surface charge)}$$

$$V(r) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho_V(r')\, dv'}{|r - r'|} \quad \text{(Volume charge)}$$

Where the primed coordinates denotes the source point location and the unprimed coordinates refer to field point (the point at which 'V' is to be determined).

→ Potential is a scalar measure of E field strength interms of the energy at that point. i.e.

$\frac{W}{Q} = V$ (volts) = potential = $\frac{\text{Work done by the charge to move to that point}}{\text{charge}}$

→ Equipotential Surface :— (Ex: Concentric sphere around the charge)

The locus of all the points whose potential difference is zero.

→ Equipotential Surface are used to graphically represent the variation of potential and its distribution.

→ $\because V = -\int \bar{E} \cdot dl$

$dv = -\bar{E} \cdot \bar{dl}$

$dv = -E\,dl \cos\theta \Rightarrow \boxed{\frac{dv}{dl} = -E\cos\theta}$ → it describes the change in 'V' with respect to the length.

When $\theta = 90°$

$V = $ constant

Equipotential Surfaces are always perpendicular Electric field lines.

When $\theta = 0°/180°$

$\left.\frac{dv}{dl}\right|_{max} = |\bar{E}|$

Conclusions :—

(1) The magnitude of Electric field intensity is the maximum rate of change of potential with distance.

(2) The direction of Electric (E) field is the direction in which potential decreases by maximum.

→ $\bar{E} = \left.-\frac{dv}{dl}\right|_{max}$

$\boxed{\bar{E} = -\nabla V}$ → potential Gradient.

→ Equipotential Surface is a Surface composed of all those points having the same value of potential. No work is involved in moving a unit charge around on an equipotential Surface, by definition, there is no potential difference between any two points on this Surface.

→ equipotential surfaces in the potential field of a point charge are spheres centered at the point charge.

→ No work is done in carrying the unit charge around any closed path, as

$$\oint E \cdot dl = 0$$

A small circle is placed on the integral sign to indicate the closed nature of the path. The above eqn is applicable for static fields.

Any field that satisfies the above eqn i.e. where the closed line integral of the field is zero, is said to be a 'conservative field'. The name arises from the fact that no work is done around a closed path.

Ex: - Gravitational field is a conservative field.

Potential Gradient & Relation between $\bar{E}$ and V :—

→ variation of potential with space is the "potential Gradient"

→ Potential is a scalar Quantity and it doesn't has a particular direction. It varies maximum in particular direction, which is called 'potential Gradient'.

→ Since

$$V = -\int E \cdot dl$$

If we apply the above eqn to a very short element of length Δl, along which $\bar{E}$ is essentially constant, leading to an increment potential difference ΔV.

$$\Delta V = -\bar{E} \cdot \Delta l$$

Consider a general region of space, in which $\bar{E}$ and V both changes as we move from point to point.

If we designate the angle between Δl and $\bar{E}$ as θ, then

$$\Delta V = -E \Delta l \cos\theta$$

$$\therefore \frac{\Delta V}{\Delta l} = -E\cos\theta$$

→ from the expression,
it is obvious that the maximum positive increment of potential, ΔV_{max} will occur when $\cos\theta = -1$ (or) Δl points the direction opposite to $\bar{E}$. for this condition,

$$\boxed{\left.\frac{dV}{dL}\right|_{max} = E} \qquad \text{(or)} \quad \nabla V = -\bar{E}$$

$$\Rightarrow \boxed{\bar{E} = -\nabla V}$$

This shows two characteristics of the relation between $\bar{E}$ and V at any point:

(1) The magnitude of the electric field intensity is given by the maximum value of the rate of change of potential with distance.

(2) This maximum value is obtained when the direction of the distance increment is opposite to $\bar{E}$ (or) The direction of $\bar{E}$ is opposite to the direction in which the potential is increasing the most rapidly.

→ for the equipotentials, $\Delta V = 0$.

→ Electric field intensity is directed from higher potential to lower potential.

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→ The gradient in the cylindrical and spherical coordinate systems follows directly from that in the cartesian system. It is noted that each term contains the partial derivative of 'v' with respect to distance in the direction of that particular unit vector.

$$\boxed{\begin{aligned} \nabla V &= \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z && \text{[cartesian]} \\ \nabla V &= \frac{\partial V}{\partial r} a_r + \frac{\partial V}{r\partial \phi} a_\phi + \frac{\partial V}{\partial z} a_z && \text{(cylindrical)} \\ \nabla V &= \frac{\partial V}{\partial r} a_r + \frac{\partial V}{r\partial \theta} a_\theta + \frac{\partial V}{r\sin\theta\,\partial \phi} a_\phi && \text{(spherical)} \end{aligned}}$$

→ ∇V is written for grad V in any coordinate system, it must be remembered that the del operator is defined only in cartesian coordinates

Potential Due to a point charge:—

We know the expression for Electric field Intensity at a distance 'r' from the point charge 'Q' is

$$E = \frac{Q}{4\pi\epsilon_0 r^2} \cdot a_r \longrightarrow ①$$

$$\text{Since } V = -\int_{-\infty}^{r} E \cdot dl \longrightarrow ②$$

The above eqn describes the energy required to move any charge particle from infinite to the distance 'r'.

∴ from ① & ②,

$$\because dl = dr \cdot a_r$$

$$V = -\int_{-\infty}^{r} \frac{Q}{4\pi\epsilon_0 r^2} \cdot a_r \cdot dr\, a_r$$

$$= -\int_{-\infty}^{r} \frac{Q}{4\pi\epsilon_0 r^2} dr = -\frac{Q}{4\pi\epsilon_0} \int_{-\infty}^{r} \frac{1}{r^2} dr = \frac{-Q}{4\pi\epsilon_0}\left[-\frac{1}{r}\right]_{-\infty}^{r}$$

$$= \frac{Q}{4\pi\epsilon_0 r}\Big]_{-\infty}^{r} = \frac{Q}{4\pi\epsilon_0 r} + 0$$

$$* \boxed{V = \frac{Q}{4\pi\epsilon_0 r}} \text{ volts.}$$

** The field inside the sphere is zero because at any radius $r_0 < r$, no charge is enclosed by this surface, so for $r_0 < r$.

$$\oint D \cdot ds = 0$$

$$\epsilon_0 \oint E \cdot ds = 0 \Rightarrow \epsilon_0 E \oint ds = 0 \Rightarrow E = 0.$$

Electric field intensity inside the sphere is zero, it needs no work to move the test charge inside and hence the potential is constant which is equal to the value at the surface of the sphere.

→ Potential due to a line charge:—

Since we know that the electric field intensity due to a line charge at a distance ρ from the line charge is

$$E = \frac{\rho_L}{2\pi\epsilon_0 \rho} a_\rho \text{ F/m}$$

and we know that   since $d\bar{l} = d\rho a_\rho$

$$V = -\int \bar{E} \cdot d\bar{l} = -\int \frac{\rho_L}{2\pi\epsilon_0 \rho} \cdot a_\rho \cdot d\rho a_\rho$$

$$\therefore V = -\int \frac{\rho_L}{2\pi\epsilon_0 \rho} \cdot d\rho = -\frac{\rho_L}{2\pi\epsilon_0} \int \frac{1}{\rho} d\rho$$

$$V = -\frac{\rho_L}{2\pi\epsilon_0} \log(\rho)$$

$$* \therefore \boxed{V = \frac{\rho_L}{2\pi\epsilon_0} \log\left(1/\rho\right)}$$

Potential varies with $\log(1/\rho)$ due to a line charge.

→ for equipotential surfaces are concentric cylinders around the charge

$\therefore$ ρ = constant $\Rightarrow$ V = constant.

→ potential due to a sheet charge:—

Since we know that the expression for electric field intensity due to a sheet charge is

$$E = \frac{\rho_s}{2\epsilon_0}.$$

Since $V = -\int E \cdot dl$

$\because dl = dx \cdot a_x$.

$$= -\int \frac{\rho_s}{2\epsilon_0} \cdot a_x \cdot dx\, a_x$$

$$V = -\frac{\rho_s}{2\epsilon_0}\int dx = \frac{-\rho_s}{2\epsilon_0} \cdot x$$

where x = constant then V = constant.

→ Equipotential surfaces are parallel to the sheet charge.

Maxwell's Equations for Electrostatic fields:—

i) Maxwell's first Equation:

Gauss's Law is the fundamental law for Maxwell's equation.

From the Gauss's Law, the total electric flux 'ψ' through any closed surface is equal to the total charge enclosed by that surface.

Thus

$$\psi = Q_{enclosed} \longrightarrow ①$$

that is,

$$\psi = \oint_S d\psi = \oint_S D \cdot ds \longrightarrow ②$$

$$= \text{total charge enclosed } Q = \int_V \rho_v\, dv \longrightarrow ③$$

(or)

$$Q = \oint_S D \cdot ds$$

By applying divergence theorem to the RHS term of the above eqn then we have

*

$$\oint_S D \cdot ds = \int_V \nabla \cdot D \, dv \longrightarrow (4)$$

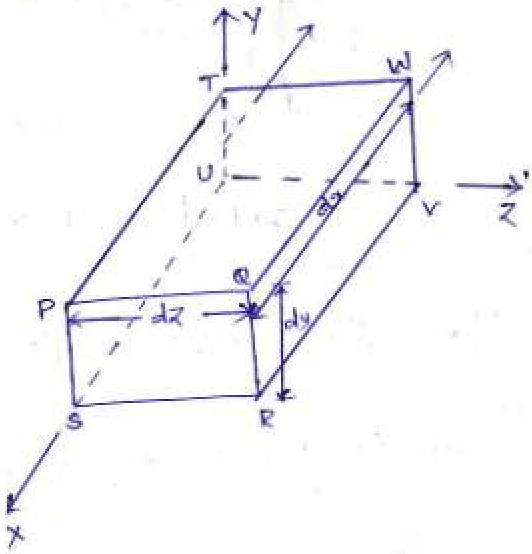
comparing two volume integrals in eqns (3) & (4), then we get

$$\int_V \nabla \cdot D \, dv = \int_V \rho_v \, dv$$

***

$$\Rightarrow \boxed{\nabla \cdot D = \rho_v}$$

Which is the first Maxwell's equation. It states that the volume charge density is the same as the divergence of the electric flux density.



Proof :—

Let us consider a region having electric flux density D, with an infinitesimal small box of sides dx, dy and dz placed at a point 'O' as shown in figure.

According to electric fields laws when there is no charge inside the box, the total electric flux entering the box is equal to the total electric flux leaving, then the box has zero divergence. On the other hand, when there exists charge density 'ρ' in the region, the flux entering may not be equal to flux leaving and so the divergence is also not zero.

Now consider the flux density D_x, D_y, D_z and following the convention that outward flux is positive, we get

flux entering the side, PQRS $= -D_x dy\, dz$

Flux leaving the side, TUVW $= \left(D_x + \frac{\partial D_y}{\partial x} \partial x\right) dy\, \partial z$

Hence the net outward flux due to D_x

$$= + \left(\cancel{D_x} + \frac{\partial D_x}{\cancel{\partial x}} \cancel{dx} \right) \partial y \partial z - D_x \cancel{\partial y} \partial z$$

$$D_x = \frac{\partial D_x}{\partial x} dx\, \partial y\, dz$$

Similarly, $D_y = \frac{\partial D_y}{\partial y} dx\,dy\,dz$

and $D_z = \frac{\partial D_z}{\partial z} \cdot dx\,dy\,dz$

Thus the total outward flux from the box $= \left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right] dx\,dy\,dz$

But the charge enclosed by the box $= \rho\, dx\,dy\,dz$

Then by Gauss's Law

$$\left[\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right]_s dx\,dy\,dz = \rho\, dx\,dy\,dz$$

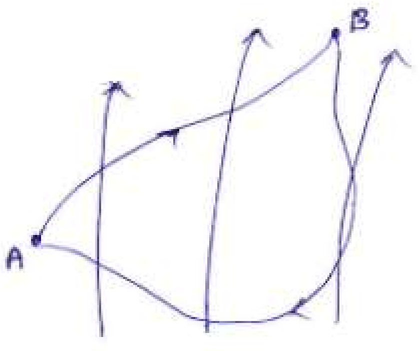
or $$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho$$

or $$\text{Div} \cdot D = \rho \qquad \left[\because \text{Div} \cdot D = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right]$$

or $$\boxed{\nabla \cdot D = \rho}$$

Maxwell's second equation for Electrostatic fields :—

Since we know that the potential difference between two point charges 'A and B' can be written as



$$V_{AB} = V_B - V_A$$

where V_B, V_A are the potentials at 'B and A' respectively. Thus the potential difference may be regarded as the potential at 'B' with reference to A.

→ the potential difference between points A and B is independent of the path taken. Hence,

$$V_{BA} = -V_{AB}$$

that is $\quad V_{AB} + V_{BA} = 0 = \oint E \cdot dl = 0$.

i.e., $\boxed{\oint_L E \cdot dl = 0} \longrightarrow (1)$

This shows that the line integral of E along a closed path as shown in figure must be zero. This implies that no net work is done in moving a charge along a closed path in an electrostatic field.

By applying stokes's theorem to the above eqn gives

$$\oint_L E \cdot dl = \int_S (\nabla \times E) \cdot ds = 0$$

(or) $\quad \boxed{\nabla \times E = 0} \longrightarrow (2)$

→ Any vector that satisfies eqn (1) or eqn (2) is said to be conservative or irrotational.

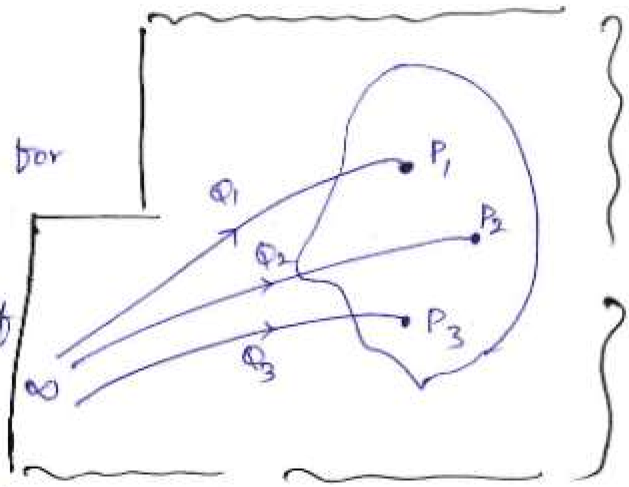
→ Vectors whose line integral doesn't depends on the path of integration are called 'conservative vectors'. Thus an Electrostatic field is a conservative field.

→ Eqn (1) or Eqn (2) is referred to as 'Maxwell's equation' (second maxwell's eqn) for static electric fields. Eqn (1) is the integral form and eqn (2) is the differential form.

Energy Density :—

To determine the energy present in an assembly of charges, we must first determine the amount of work necessary to assemble them. Bringing a positive charge from infinity into the field of another positive charge requires work the work being done by the external source moving the charge. Let us imagine that the external source carries the charge upto a point near the fixed charge and then holds it there. Energy must be conserved.

Suppose we wish to move the position of three point charges Q_1, Q_2 and Q_3 into an empty space from infinity. Initially the space has no charge i.e. charge free as shown in figure.



Bringing a charge Q_1 from infinity to any position requires no work, for there is no field present. The position of Q_2 at a point in the field of Q_1 requires an amount of work given by the product of the charge Q_2 and the potential at that point due to Q_1. We represent this potential as V_{21}, where the first subscript indicates the location [source] and the second subscript indicates the location towards Q_1 [destination]. That is, V_{21} is the potential at the location of Q_2 due to Q_1. Then

$$\text{work done to move the position of } (Q_2) = Q_2 V_{21}$$

Similarly, we may express the work required to move the position of each additional charge in the field of all those already present.

Work done to move the position of $Q_3 = Q_3 V_{31} + Q_3 V_{32}$

work done to move the position of $Q_4 = Q_4 V_{41} + Q_4 V_{42} + Q_4 V_{43}$

and so forth.

→ The total work is obtained by adding each contribution:

$$W_E = W_1 + W_2 + W_3 + W_4 + \cdots$$

$$W_E = 0 + Q_2 V_{21} + Q_3 V_{31} + Q_3 V_{32} + Q_4 V_{41} + Q_4 V_{42} + Q_4 V_{43} + \cdots$$

$$= Q_2(V_{21}) + Q_3(V_{31} + V_{32}) + Q_4(V_{41} + V_{42} + V_{43}) + \cdots \longrightarrow (1)$$

Here $Q_2 V_{21} = Q_2 \cdot \frac{Q_1}{4\pi\epsilon_0 R_{12}}$

$Q_3 V_{31} = Q_3 \cdot \frac{Q_1}{4\pi\epsilon_0 R_{13}}$, $Q_3 V_{32} = Q_3 \cdot \frac{Q_2}{4\pi\epsilon_0 R_{23}}$

⋮

where $R_{12}, R_{13}, R_{23} \cdots$ are represents the scalar distance b/w Q_1 and Q_2, Q_1 and Q_3, Q_2 and Q_3 respectively.

If the charges were positioned in reverse order, then the expression for total work is

$$W_E = Q_1 V_{12} + Q_1 V_{13} + Q_2 V_{23} + Q_1 V_{14} + Q_2 V_{24} + Q_3 V_{34} + \cdots \longrightarrow (2)$$

Adding the two energy expressions (1) and (2) gives

$$2W_E = Q_1(V_{12} + V_{13} + V_{14} + \cdots) + Q_2(V_{21} + V_{23} + V_{24} + \cdots) + Q_3(V_{31} + V_{32} + V_{34} + \cdots)$$
$$+ Q_4(V_{41} + V_{42} + V_{43} + V_{45} + \cdots) + \cdots$$

Each sum of potentials in parentheses is the combined potential due to all the charges except for the charge at the point where this combined potential is being found. In other words,

$$V_{12} + V_{13} + V_{14} + \cdots = V_1$$

the potential at the location of Q_1 due to the presence of $Q_2, Q_3, \cdots$

Therefore we have,

$$W_E = \frac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3 + \cdots) = \frac{1}{2}\sum_{m=1}^{N} Q_m V_m \quad \text{(in Joules)} \longrightarrow (3)$$

To obtain an expression for the energy stored in a region of continuous charge distribution, each charge is replaced by $\rho_v dv$ (or) $\rho_L dl$ (or) $\rho_s ds$ and the summation becomes an integral. i.e.,

$$W_E = \frac{1}{2}\int_{vol} \rho_v V dv \longrightarrow 4(i) \quad \text{(or)} \quad W_E = \frac{1}{2}\int_{L} \rho_L V dl \longrightarrow 4(ii) \quad \text{(or)} \quad W_E = \frac{1}{2}\int_{S} \rho_s V ds \longrightarrow 4(iii)$$

Since

$\rho_v = \nabla\cdot D$, so eqn 4(i) can be written as

$$W_E = \frac{1}{2}\int_{V} (\nabla\cdot D) V dv \longrightarrow (5)$$

But for any vector $\bar{A}$ and scalar V, the identity

$$\nabla\cdot VA = A\cdot\nabla V + V(\nabla\cdot A)$$

$$\text{(or)} \quad (\nabla\cdot A)V = \nabla\cdot VA - A\cdot\nabla V \longrightarrow (6)$$

Applying the identity in eqn (6) to eqn (5), we get

$$W_E = \frac{1}{2}\int_{V} (\nabla\cdot VD)dv - \frac{1}{2}\int_{V} (D\cdot\nabla V)dv \longrightarrow (7)$$

By applying divergence theorem to the first term on the right hand-side of this equation, we have

$$W_E = \frac{1}{2}\oint_{S} (VD)\cdot ds - \frac{1}{2}\int (D\cdot\nabla V)dv \longrightarrow (8)$$

Since V (potential) varies as $1/r$ and D as $1/r^2$ for point charges. V varies as $1/r^2$ and D as $1/r^3$ for dipoles and so on. Hence, VD in the first term on the right side of eqn (8) must vary atleast as $1/r^3$ while ds varies as r^2. Consequently, the first integral in eqn (8) must tend to zero as the Surface 'S' becomes large. Hence, eqn (8) reduces to

$$W_E = -\frac{1}{2}\int_V (D\cdot\nabla V)\,dv = \frac{1}{2}\int_V (D\cdot E)\,dv \longrightarrow (9)$$

and we know $E = -\nabla V$ and $D = \epsilon_0 E$

$$\therefore W_E = \frac{1}{2}\int_V D\cdot E\,dv = \frac{1}{2}\int_V \epsilon_0 E^2\,dv \longrightarrow (10)$$

from this, we can define electrostatic energy density w_E (in J/m^3) as

$$w_E = \frac{dW_E}{dv}$$

from eqn (10),

$$dW_E = \frac{1}{2} D\cdot E\,dv.$$

$$\frac{dW_E}{dv} = \frac{1}{2} D\cdot E = \frac{1}{2}\epsilon_0 E^2 = \frac{D^2}{2\epsilon_0}.$$

$\therefore$ Energy density is

$$w_E = \frac{dW_E}{dv} = \frac{1}{2}\epsilon_0 E^2 \pm \frac{D^2}{2\epsilon_0}.$$

we can write it as

$$\therefore \boxed{W_E = \int_V w_E\,dv}$$

Electrostatics – II

→ In this, we study about the theory of electric phenomena in material space.

→ Materials are broadly classified interms of their electrical properties as conductors and Non-conductors.

→ Non-conducting materials are usually referred to as "Insulators (or) Dielectrics".

→ Materials may be classified interms of their conductivity σ in mhos per meter ($\mho$/m) or Siemens per meter (S/m) as conductors and non conductors or technically as metals and insulators (or dielectrics).

→ The conductivity of material depends on temperature and frequency.

→ A material with high conductivity ($\sigma >> 1$) is referred to as a Metal, whereas one with low conductivity ($\sigma << 1$) is referred to as an Insulator.

→ A material whose conductivity lies some where between those of metals and insulators is called a semiconductor.

→ Examples of materials such as metals are copper and Aluminium. Semiconducting materials are Silicon and Germanium and Glass and rubber are insulators.

→ The conductivity of metals generally increases with decrease in temperature.

→ At temperatures near absolute zero ($T = 0\,K$), some conductors exhibit infinite conductivity and are called "super conductors".

→ Lead and Aluminium are examples of super conductors. The conductivity of Lead at 4K is of the order of 10^{20} S/m.

→ The major difference between a metal and an insulator lies in the number of electrons available for conduction of current.

→ Dielectric materials have few electrons available for conduction of current, whereas metals have an abundance of free electrons.

Convection and conduction Currents :—

→ Electrical voltage (or potential difference) and Current are two fundamental quantities in Electrical Engineering.

→ Electric Current is Caused by the motion of electric charges.

→ The Current (in amperes) through a given area is the electric charge passing through the area per unit time. That is,

$$I = \frac{dQ}{dt} \longrightarrow ①$$

Thus in a Current of one ampere, charge is being transferred at a rate of one Coulomb per second.

→ If Current ΔI flows through a planar Surface ΔS, the Current density is

$$J = \frac{\Delta I}{\Delta S}$$

$$\Delta I = J \Delta S \longrightarrow ②$$

assuming that the Current density is perpendicular to the Surface.

→ If the Current density is not normal to the Surface,

$$\Delta I = J \cdot \Delta S \longrightarrow ③$$

Thus, the total Current flowing through a Surface 'S' is

$$I = \int_S J \cdot ds \longrightarrow ④$$

The above eqn applies to any kind [i.e, convection and conduction] of Current densities .

ETTL U-II (2)

→ Convection Current, as distinct from conduction Current, doesnot involve conductors and consequently doesnot satisfy ohm's law. ** Convection Current is a flow of current charges due to the absence of an Electric field. It is produced by a beam of e's flowing through an insulating medium.

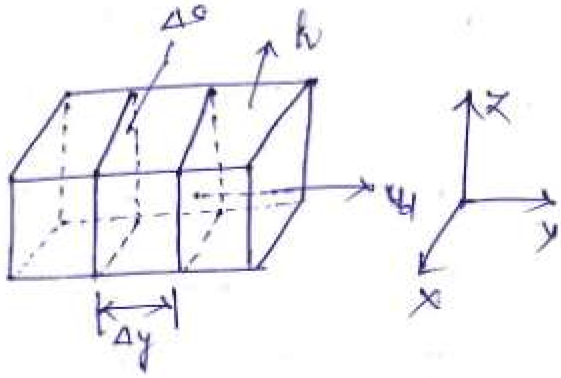
→ Convection Current occurs when current flows through an insulating medium such as Liquid, rarefied gas or a Vacuum.

→ A beam of electrons in a Vacuum tube causes a Convection Current.

→ consider a filament of figure following. If there is a flow of charge of density ρ_v, at velocity v_d along a_y direction. from eqn (1) The Current through the filament is

$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \, \Delta s \frac{\Delta y}{\Delta t} a_y$$

$$= \rho_v \, \Delta s \, v_d a_y \longrightarrow (5)$$



The Current density at a given point is the Current through a unit normal area at that point.

The y-directed Current density J_y is given by

$$J_y = \frac{\Delta I}{\Delta s} = \rho_v v_d \longrightarrow (6)$$

Hence, in general

$$J = \rho_v v_d \longrightarrow (7)$$

The Current I is the Convection Current and J is the Convection Current density in Amperes per square Meter (A/m^2).

→ conduction Current is a flow of charges due to the presence of an Electric field. it is the Current produced due to the flow of electrons in a conductor. This obeys ohms Law.

→ Conduction Current requires a conductor. A conductor is characterized by a large number of free electrons that provide conduction current due to an impressed electric field. When an electric field $\vec{E}$ is applied, the force on an electron with charge $-e$ is

$$f = -eE \longrightarrow (8)$$

Since the electron is not in free space, it will not experience an average acceleration under the influence of the electric field. It suffers constant collisions with the atomic lattice and drifts from one atom to another.

→ If an electron with mass 'm' is moving in an electric field $\vec{E}$ with an average drift velocity v_d, according to Newton's Law, the average change in momentum of the free electron must match the applied force. Thus,

$$\frac{m v_d}{\tau} = -eE$$

(or) $$v_d = -\frac{e\tau}{m} E$$

where 'τ' is the average time interval between collisions. This indicates that the drift velocity of the electron is directly proportional to the applied field.

→ If there are 'n' electrons per unit volume, the electronic charge density is given by

$$\rho_v = -ne$$

Thus the conduction current density is,

$$J = \rho_v v_d = \frac{ne^2\tau}{m} E = \sigma E$$

(or)

$$J = \sigma E$$ where $\sigma = \frac{ne^2\tau}{m}$ is the conductivity of the conductor

ETTL
U-III (3)

Polarization in Dielectrics:—

→ The main difference between a conductor and a dielectric lies in the availability of free electrons in the outermost atomic shells to conduct current. Although the charges in a dielectric are not able to move freely, they are bound by finite forces and we may certainly expect a displacement when external force is applied.

→ To understand the effect of an Electric field on a dielectric, consider an atom of the dielectric as consisting of a negative charge $-Q$ and a positive charge $+Q$ as in figure below. Since the atom or molecule is electrically neutral i.e., the atom has equal amounts of positive and negative charge. When an Electric field E is applied, the positive charge is displaced from its equilibrium position in the direction of E by the force $F_+ = QE$, while the negative charge is displaced in the opposite direction by the force $F_- = QE$. A dipole results from the displacement of the charges, and the dielectric is said to be polarized. This distorted charge distribution is equivalent to the original distribution plus a dipole whose moment is

$$\boxed{P = Qd} \longrightarrow (1)$$

where d is the distance vector from $-Q$ to $+Q$ of the dipole.

→ If there are N dipoles in a volume Δv of the dielectric, the total dipole moment due to the electric field is

$$\boxed{Q_1 d_1 + Q_2 d_2 + \cdots + Q_N d_N = \sum_{k=1}^{N} Q_k d_k} \left(= P_{tot}\right) \longrightarrow (2)$$

As a measure of intensity of the polarization, we define polarization P as the dipole moment per unit volume of the dielectric; that is,

$$\boxed{P = \lim_{\Delta v \to 0} \frac{P_{tot}}{\Delta v} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{N} Q_k d_k}{\Delta v}} \longrightarrow (3)$$

Thus we conclude that the major effect of the E-field on a dielectric is the creation of dipole moments that align themselves in the direction of $\bar{E}$. This type of dielectric is said to be "Non polar". Examples of such dielectrics are Hydrogen, oxygen, Nitrogen and rare gases.

** Non-polar dielectric molecules do not possess dipoles untill the application of the Electric field.

** polar Dielectric molecules have built-in permanent dipoles, those are randomly oriented and are said to be 'polar'. When an electric field $\bar{E}$ is applied to a polar molecule, the permanent dipole experiences a torque tending to align its dipole moment parallel with $\bar{E}$.

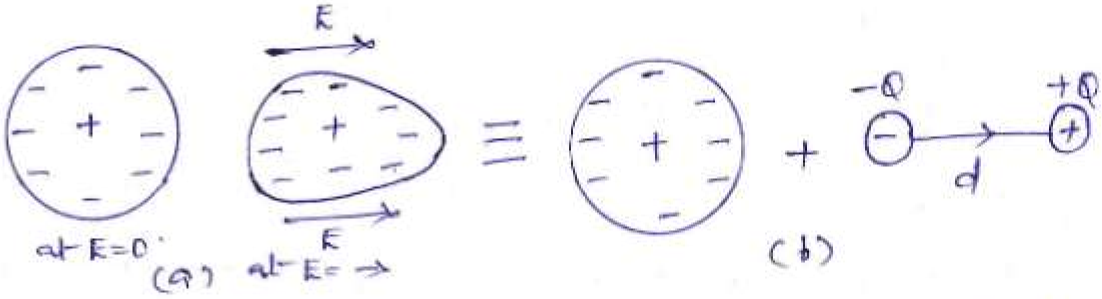
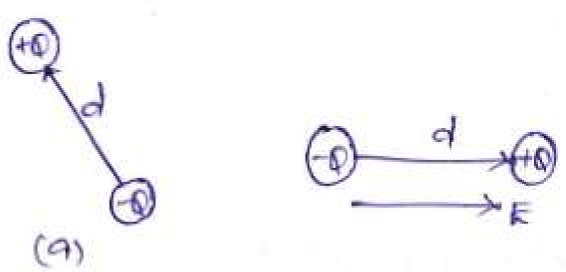


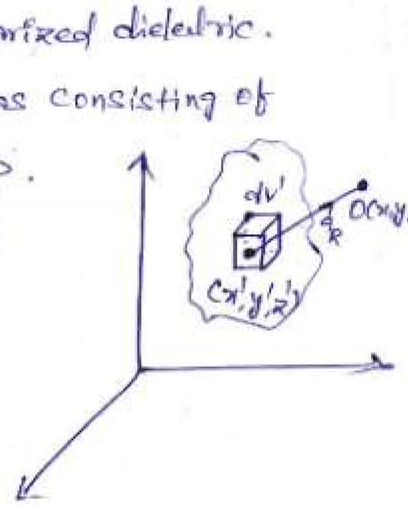
fig: polarization of a non-polar atom or molecule



(A) permanent dipole ($E=0$)
(b) Alignment of permanent dipole ($E\neq 0$)

fig: polarization of a polar molecule.

→ Let us now calculate the field due to a polarized dielectric. consider the dielectric material shown in figure, as consisting of dipoles with dipole moment $\bar{P}$ per unit volume is P. The potential dv at an exterior point O due to the dipole moment $P dv'$ is



$$dv = \frac{P \cdot a_R \, dv'}{4\pi\epsilon_0 R^2} \longrightarrow (1)$$

where $R^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$ and $\bar{R}$ is the distance between the volume element dv' at (x', y', z') and the field point $O(x,y,z)$.

Dielectric constant:—

→ The gradient of $1/R$ with respect to the primed coordinates is

$$\nabla'(1/R) = \frac{a_R}{R^2} \longrightarrow (5)$$

where ∇' is the del operator with respect to (x', y', z'). Thus

$$\frac{P \cdot a_R}{R^2} = P \cdot \nabla'(1/R) \longrightarrow (6)$$

Applying the vector identity $\nabla' \cdot fA = f\nabla' \cdot A + A \cdot \nabla' f$

$$\frac{P \cdot a_R}{R^2} = \nabla' \cdot (P/R) - \frac{\nabla' \cdot P}{R} \longrightarrow (7)$$

from the above dv expression, we can write

$$\therefore V = \int_V \frac{1}{4\pi\epsilon_0}\left[\nabla' \cdot \frac{P}{R} - \frac{1}{R}\nabla' \cdot P\right] dv'$$

By applying Divergence theorem to the first term leads finally to

$$V = \int_{S'} \frac{P \cdot a_n'}{4\pi\epsilon_0 R} ds' + \int_{V'} \frac{-\nabla' \cdot P}{4\pi\epsilon_0 R} dv' \longrightarrow (8)$$

where a_n' → outward unit normal to surface ds' of the dielectric.

since we have

$$V(r) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\rho_s(r')\, ds'}{|r - r'|} \quad \text{(Surface charge)} \longrightarrow 9(a)$$

$$V(r) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho_v(r')\, dv'}{|r - r'|} \quad \text{(Volume charge)} \longrightarrow 9(b)$$

$\Big\} \longrightarrow (9)$

By comparing the two terms on the right hand side of eqn (8) with eqn (9), show that the two terms denote the potential due to surface and volume charge distributions with densities.

$$\boxed{\begin{aligned} \rho_{ps} &= P \cdot a_n \\ \rho_{pv} &= -\nabla \cdot P \end{aligned}} \longrightarrow (10)$$

→ If we consider the case in which the dielectric region contains free charge. If ρ_v is the volume density of free charge, the total volume charge density ρ_t is given by

$$\rho_t = \rho_v + \rho_{pv} = \nabla \cdot \epsilon_0 E$$

Hence $\rho_v = \nabla \cdot \epsilon_0 E - \rho_{pv}$

$= \nabla \cdot (\epsilon_0 E + P)$ Since $\rho_{pv} = -\nabla \cdot P$

$$\boxed{\rho_v = \nabla \cdot D}$$

where

$$\boxed{D = \epsilon_0 E + P}$$

We conclude that, the net effect on the dielectric due to the electric field E is to increases D inside it by the amount P.

→ The application of E to the dielectric material causes the flux density to be greater than that would be in free space.

→ for free space, $P=0$ i.e. $D = \epsilon_0 E$.

→ for some dielectrics, P is proportional to the applied Electric field E, and we have

$$\boxed{P = \chi_e \epsilon_0 E}$$

where χ_e (chi) known as the Electric susceptibility of the material and it is a dimensionless quantity.

Dielectric constant :-

→ Due to the applied Electric field E, the electric flux density D in a dielectric material is

$$D = \epsilon_0 E + P$$

since $P = \chi_e \epsilon_0 E$

$$\therefore \boxed{D = \epsilon_0 [1 + \chi_e] E} \longrightarrow (1)$$

Since we have $\boxed{D = \epsilon E}$ → (2)

where $\epsilon = \epsilon_0 \epsilon_r$

$\epsilon \rightarrow$ permittivity of the dielectric
$\epsilon_0 \rightarrow$ permittivity of free space.
$\epsilon_r \rightarrow$ Relative permittivity or Dielectric constant

$\therefore \quad D = \epsilon_0 \epsilon_r E$ → (3)

from eqns (1) & (3), we can write

$$\boxed{\epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0}} \rightarrow (4)$$

** The dielectric constant (or relative permittivity) ϵ_r is the ratio of the permittivity of the dielectric to that of free space.

** ϵ_r and χ_e are dimensionless and ϵ and ϵ_0 are in farads per meter.

→ for free space and Non-dielectric materials (such as metals) $\epsilon_r = 1$.

** Practically, no dielectric is ideal. when the E-field in a dielectric is sufficiently large, it begins to pull electrons completely out of the molecules, and the dielectric becomes conducting, Dielectric breakdown may occur.

→ Dielectric breakdown means change of properties of non-conducting material to conducting properties.

→ Dielectric breakdown occurs in all kinds of dielectric materials (gases, Liquids or solids) and it depends on the nature of the material, temperature, humidity and the amount of time that the field is applied.

→ The minimum value of the Electric field at which dielectric break-down occurs is called the "dielectric strength of the dielectric Material".

Isotropic and Homogeneous Dielectrics :—

→ Materials for which ϵ (or σ) doesnot vary in the region being considered and is therefore the same at all points (i.e., independent of x, y, z) are said to be homogeneous.

→ The materials are said to be inhomogeneous (or Non-homogeneous) when ϵ is dependent on the space coordinates.

→ Isotropic dielectrics are those that have the same properties in all directions.

→ A dielectric material (in which $D = \epsilon E$ applies) is linear if ϵ (permittivity of dielectric material) doesnot change from point to point and isotropic if ϵ doesnot change with the direction.

→

→ Incase of conductors: The material is said to be homogeneous if σ is the same at all points and isotropic if σ doesnot vary with direction. [$J = \sigma E$ applies].

ETTL U-II ⑥

Continuity Equation :—

→ from the principle of conservation of charge, the time rate of decrease of charge within a given volume must be equal to the net outward current flows through the surface of the volume. Thus the current (I_{out}) coming out of the closed surface is

$$I_{out} = \oint_S J \cdot ds = \frac{-dQ}{dt} \longrightarrow ①$$

where Q is the total charge enclosed by the closed surface.

from Divergence theorem,

$$\oint_S J \cdot ds = \int_V \nabla \cdot J \, dv \longrightarrow ②$$

But

$$Q = \int_V \rho_v \, dv \longrightarrow ③$$

from ① & ③,

$$\oint_S J \cdot ds = -\frac{d}{dt}\int_V \rho_v dv = -\int_V \frac{\partial \rho_v}{\partial t} dv \longrightarrow ④$$

from eqns ② & ④,

$$\int_V \nabla \cdot J \, dv = -\int_V \frac{\partial \rho_v}{\partial t} dv$$

***

$$\therefore \boxed{\nabla \cdot J = -\frac{\partial \rho_v}{\partial t}}$$ → continuity of current equation (or) continuity equation.

→ The continuity equation states that there can be no accumulation of charge at any point.

→ for steady currents i.e., total charge leaving a volume is equal to the total charge entering into the volume. so the net charge present in the volume is same. Therefore the charge density ρ_v is same

$\therefore\ \rho_v = \text{constant} \Rightarrow \frac{\partial \rho_v}{\partial t} = 0$

from the continuity Equation,

** $\boxed{\nabla \cdot J = 0}$ → for steady currents.

Relaxation Time :—

To consider the effect of introducing charge at some interior point of a given material (conductor or Dielectric). we make use of continuity Equation in conjuction with ohm's Law.

from ohms Law, $J = \sigma E$ → ①

and from Gauss's Law, $\nabla \cdot D = \rho_v$

for Homogeneous material, ϵ is same i.e, constant.

so $\nabla \cdot \epsilon E = \rho_v$

$$\nabla \cdot E = \frac{\rho_v}{\epsilon} \longrightarrow ②$$

Multiply Eqn② with σ on both the sides. we get

$$\sigma\, \nabla \cdot E = \sigma \frac{\rho_v}{\epsilon} \longrightarrow ③$$

since the material is homogeneous, so σ is constant.

$\therefore$ Eqn③ can be written as

$$\nabla \cdot \sigma E = \frac{\sigma}{\epsilon} \rho_v \longrightarrow ④$$

from ① & ④, $\nabla \cdot J = \frac{\sigma}{\epsilon} \rho_v \longrightarrow ⑤$

since continuity Eqn is $\nabla \cdot J = -\frac{\partial \rho_v}{\partial t} \longrightarrow ⑥$

from ⑤ & ⑥,

$$\frac{-\partial \rho_v}{\partial t} = \frac{\sigma}{\epsilon} \rho_v \Rightarrow \frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\epsilon} \rho_v = 0 \longrightarrow ⑦$$

This is a homogeneous linear ordinary differential Eqn. This can be

Solved by using variable separable method. i.e.,

$$\frac{\partial \rho_v}{\partial t} = -\frac{\sigma}{\epsilon}\rho_v \Rightarrow \frac{\partial \rho_v}{\rho_v} = -\frac{\sigma}{\epsilon}\partial t$$

By integrating on both the sides

$$\ln(\rho_v) = -\frac{\sigma t}{\epsilon} + \ln(\rho_{vo}) \longrightarrow (8)$$

where $\ln(\rho_{vo})$ is a constant of integration. Thus

$$\boxed{\rho_v = \rho_{vo}\, e^{-t/T_r}} \longrightarrow (9)$$

where ** $\boxed{T_r = \frac{\epsilon}{\sigma}}$ seconds $\longrightarrow (10)$

T_r is the time constant in seconds.

From eqn (9), ρ_{vo} is the initial charge density (i.e., ρ_v at $t=0$). The equation shows that the introduction of charge at some interior point of the material results in a decay of volume charge density (ρ_v) exponentially.

→ The time constant 'T_r' is known as the 'Relaxation Time' or 'Rearrangement time'.

** Relaxation Time is the time taken by a charge placed in the interior of a material to drop its charge density to $\frac{1}{e}$ (=36.8%) times of its initial value.

→ Relaxation time is less for Good Conductors and more for Good Dielectrics.

→ For Example, for copper $\sigma = 5.8\times10^{7}$ S/m, $\epsilon_r = 1$ therefore

$$T_r = \frac{\epsilon_0\epsilon_r}{\sigma} = \frac{1\times10^{-9}}{36\pi}\times\frac{1}{5.8\times10^{7}} = 1.53\times10^{-19}\text{ sec.}$$

Showing that a rapid decay of charge placed inside copper. This implies that for good conductors, the relaxation time is so short that most of the charge will vanish from any interior point and appear at the surface (as surface charge) almost instantaneously.

→ for fused Quartz, $\sigma = 10^{-17}$ S/m, $\epsilon_r = 5.0$.

$$T_r = \frac{5\times 10^{-9}}{36\pi} \times \frac{1}{10^{-17}} = 51.2 \text{ days}$$

showing a very large relaxation time. Thus for good dielectrics, one may consider the introduced charge to remain wherever placed for times upto days.

→ The electrons which are introduced to good conductor vanishes quickly.

Poisson's and Laplace's Equations :—

Poisson's and Laplace's Equations are used to solve Electro-static problems involving a set of conductors maintained at different potentials.

→ Laplace's Equation is used in deriving the resistance of an object and the capacitance of a capacitor.

→ Poisson's and Laplace's equations are easily derived from Gauss's law (for a linear, isotropic material medium).

from Gauss's law, $\nabla \cdot D = \rho_v \longrightarrow (1)$

Since we have, $E = -\nabla V \longrightarrow (2)$

and

$D = \epsilon E \longrightarrow (3)$

from (1) & (3), $\nabla \cdot \epsilon E = \rho_v \longrightarrow (4)$

from eqns (2) & (4), $\nabla \cdot (\epsilon . -\nabla V) = \rho_v$

$$\boxed{\nabla \cdot (-\epsilon \nabla V) = \rho_v} \longrightarrow (5)$$

Since it is homogeneous medium, so ϵ is constant.

$\therefore$ from eqn (5), $\nabla \cdot \nabla V = -\frac{\rho_v}{\epsilon}$

**

$$\Rightarrow \boxed{\nabla^2 V = -\frac{\rho_v}{\epsilon}} \longrightarrow (6)$$

The above eqn is called 'poisson's Equation' for homogeneous medium.

→ If the medium is not homogeneous, then ϵ is not same at all points so

$$\boxed{\nabla \cdot (-\epsilon \nabla V) = +\rho_v} \longrightarrow (7)$$

is the poisson's Equation for inhomogeneous medium.

→ If the medium is charge free medium then $(\rho_v = 0)$

$$\boxed{\nabla^2 V = 0} \longrightarrow (8)$$

Laplace's Equation.

where $\nabla^2 \rightarrow$ Laplacian operator

→ The Laplace's Equation in cartesian, Cylindrical or spherical coordinates respectively is given by

cartesian ← $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$ ⟶ (9)

cylindrical ← $\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$ ⟶ (10)

spherical ← $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$ ⟶ (11)

General formula is

***

$$\boxed{\nabla^2 V = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_3 h_1}{h_2}\frac{\partial V}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial w}\right)\right]}$$

*

Where

u v w	h_1 h_2 h_3
x y z	1 1 1
ρ ϕ z	1 ρ 1
r θ ϕ	1 r $r\sin\theta$

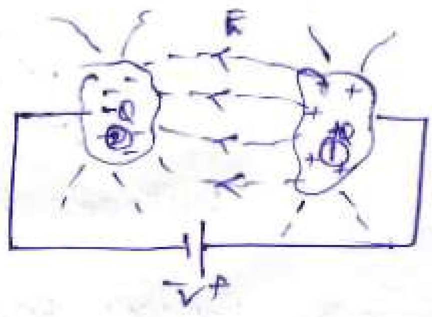
→ Depending on the coordinate variables used to express V, that is V(x,y,z), V(ρ,φ,z) or V(r,θ,φ). poisson's equations in those coordinate systems may be obtained by simply replacing zero on the right hand side of eqns ⑨ ⑩ & ⑪ with $-\rho_v/\epsilon$.

Capacitance :—

→ Any two conducting bodies separated by free space or a dielectric material exhibits a capacitance between them.

→ Generally, to have a capacitor we must have two (or more) conductors carrying equal but opposite charges. This implies that all the flux lines leaving one conductor must necessarily terminate at the surface of the other conductor. The conductors are sometimes referred to as the plates of the capacitors. The plates may be separated by free space or dielectric.

→ consider the two conductor capacitor of figure. The conductors are maintained at a potential difference V given by



$$V = V_1 - V_2 = -\int_2^1 E \cdot dl$$

where 'E' is the electric field existing between the conductors and conductor 1 is assumed to carry a positive charge.

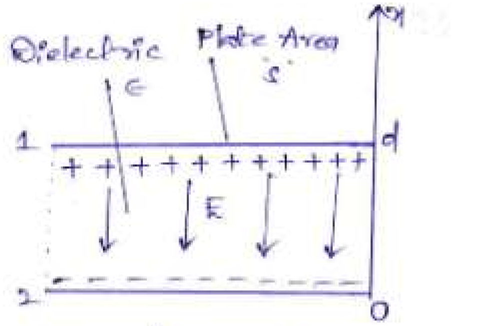
→ The capacitance 'C' of a capacitor is the ratio of the magnitude of the charge on one of the plates to the potential difference between them; that is

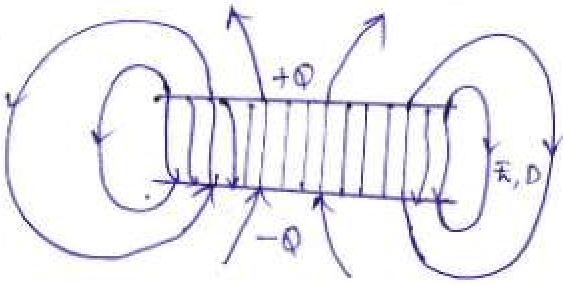
$$\boxed{C = \frac{Q}{V} = \frac{\epsilon \oint E \cdot dl}{\int E \cdot dl}}$$

- The negative sign before $V = -\int E \cdot dl$ has been dropped because we are interested in the absolute value of 'V'.

→ The capacitance 'C' is a physical property of the capacitor and is measured in farads.

Parallel plate capacitor :—





Consider the parallel plate capacitor. Suppose that each of the plates has an area 'A' and they are separated by a distance 'd'. We assume that plates 1 and 2 respectively carry charges +Q and −Q uniformly distributed on them so that

$$\rho_s = \frac{Q}{A} \longrightarrow (1)$$

An ideal parallel-plate capacitor is one in which the plate separation 'd' is very small compared with the dimensions of the plate. Assuming such an ideal case, the fringing field at the edge of the plates shown in figure can be ignored so that the field between them is considered uniform. If the space between the plates is filled with a homogeneous dielectric with permittivity 'ϵ' and we ignore flux fringing at the edges of the plates.

$$E = \frac{-\rho_s}{\epsilon} a_x \longrightarrow (2)$$

[x direction is reverse]

Since $\rho_s = \frac{Q}{A}$

$$\therefore E = -\frac{Q}{\epsilon A} a_x \longrightarrow (3)$$

Since we have a formula for 'V' in terms of 'E' i.e.,

$$V = -\int E \cdot dl \longrightarrow (4)$$

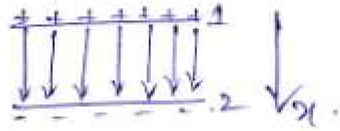
$$\therefore V = -\int_0^d \left[\frac{-Q}{\epsilon A} a_x \cdot dx\, a_x\right] = \frac{Q}{\epsilon A}\int_0^d dx$$

$$\therefore V = \frac{Q}{\epsilon A}(x)_0^d = \frac{Qd}{\epsilon A} \longrightarrow (5)$$

Since we have a formula for capacitance,

$$\boxed{C = \frac{Q}{V} = \frac{\epsilon A}{d}} \quad ***$$

* ———— *



Electric field lines always starts at +ve charge and terminates at −ve charge

→ Generally, E due to sheet is normal to sheet on either side of the sheet.

Due to conductor 1:

$E = \frac{+\rho_s}{2\epsilon_0} a_x$ (RHS)

$E = \frac{-\rho_s}{2\epsilon_0} a_x$ (LHS)

Due to conductor 2:

$E = \frac{-\rho_s}{2\epsilon_0} a_x$ (RHS)

$E = \frac{+\rho_s}{2\epsilon_0} a_x$ (LHS)

In b/w plates means, we consider RHS of conductor (1) & LHS of conductor (2)

$$E = \frac{\rho_s}{\epsilon} a_x.$$

where $\epsilon \rightarrow$ permittivity of dielectric

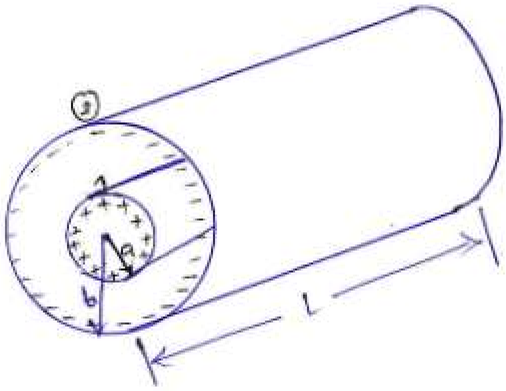
$A \rightarrow$ Area of the plates

$d \rightarrow$ separation of plates.

The above formula offers a means of measuring the dielectric constant ϵ_r of a given dielectric. By measuring the capacitance C of a parallel-plate capacitor with the space between the plates filled with the dielectric and the capacitance C_0 with air between the plates. we find

from $\boxed{\epsilon_r = \dfrac{C}{C_0}}$

coaxial capacitor :—



→ A coaxial capacitor is essentially a coaxial cable or coaxial cyllindrical capacitor. consider length 'L' of two coaxial conductors of inner radius 'a' and outer radius 'b' $(b>a)$ as shown in figure. Let the space between the conductors be filled with a homogeneous dielectric with permittivity ϵ. we assume that conductors ① and ② respectively carry $+Q$ and $-Q$ uniformly distributed on them. By applying Gauss's law to an arbitrary Gaussian cyllindrical surface of radius ρ $(a<\rho<b)$, we obtain

$$Q = \epsilon \oint E \cdot ds = \epsilon E_\rho 2\pi \rho L$$

Hence $$E = \frac{Q}{2\pi \epsilon \rho L} a_\rho$$

Neglecting flux fringing at the cyllinder ends,

$$V = -\int_b^a E \cdot dl = -\int_b^a \left[\frac{Q}{2\pi\epsilon\rho L} a_\rho\right] \cdot d\rho\, a_\rho = \frac{-Q}{2\pi\epsilon L}\int_b^a \left(\frac{1}{\rho}\right) d\rho$$

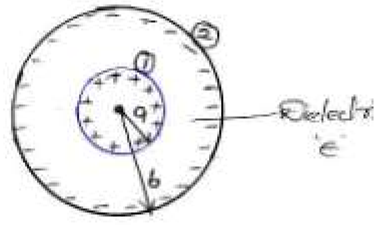
$$V = \frac{-Q}{2\pi\epsilon L} \ln(\rho)\Big]_b^a = \frac{-Q}{2\pi\epsilon L}\ln(a/b) = \frac{Q}{2\pi\epsilon L}\ln(b/a)$$

***

Since capacitance $C = \dfrac{Q}{V} \Rightarrow \boxed{C = \dfrac{2\pi\epsilon L}{\ln(b/a)}}$

Spherical Capacitor:—

→ A spherical capacitor is the case of two concentric spherical conductors. Consider the inner sphere of radius 'a' and outer sphere of radius b (b>a) separated by a dielectric medium with permittivity 'ε' as shown in figure. We assume charges +Q and −Q on the inner and outer spheres respectively.



By applying Gauss's Law to an arbitrary Gaussian spherical surface of radius r (a<r<b), we have

$$Q = \epsilon \oint E \cdot ds = \epsilon E_r 4\pi r^2$$

$$\left\{ \because \oint ds = \text{Area of the sphere} = 4\pi r^2 \right\}$$

that is, $E = \frac{Q}{4\pi\epsilon r^2} a_r$

The potential difference between the conductors is

$$V = -\int_b^a E \cdot dl = -\int_b^a \left[\frac{Q}{4\pi\epsilon r^2} a_r\right] \cdot dr\, a_r$$

$$V = \frac{-Q}{4\pi\epsilon}\int_b^a \frac{1}{r^2}\, dr = \frac{-Q}{4\pi\epsilon}\left[-1/r\right]_b^a$$

$$V = \frac{Q}{4\pi\epsilon}\left[1/a - 1/b\right]$$

Thus the capacitance of the spherical capacitor is

***

$$\boxed{C = \frac{Q}{V} = \frac{4\pi\epsilon}{\left[1/a - 1/b\right]}} \text{ farads}$$

→ if $b \to \infty$, $C = 4\pi\epsilon a$, which is the capacitance of a spherical capacitor whose outer plate is infinitely large. Such is the case of a spherical conductor at a large distance from other conducting bodies — the isolated sphere.

Energy stored in a Capacitor:—

Since we have a formula for Energy $E = \frac{1}{2}\int \epsilon E^2 dv$

Assume the Area of the capacitor plate is 'A' and the separation between two plates is 'd' then the total closed area (or) volume of capacitor is A.d.

$\therefore W_E = \frac{1}{2}\epsilon E^2 \int dv = \frac{1}{2}\epsilon E^2 Ad$

since $E = \frac{V}{d}$

$\therefore W_E = \frac{1}{2}\epsilon \left(\frac{V}{d}\right)^2 Ad$

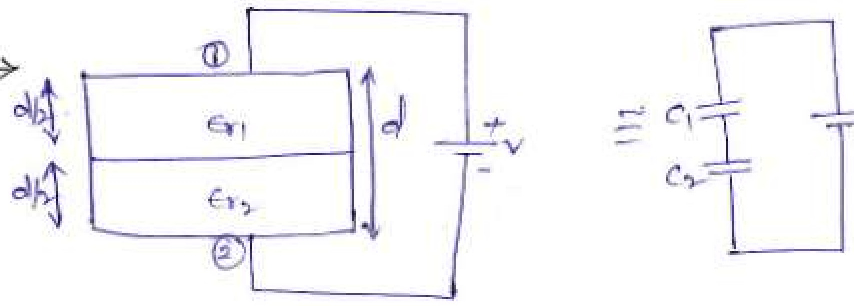
$W_E = \frac{1}{2}\frac{\epsilon V^2}{d^2}Ad = \frac{1}{2}\frac{\epsilon A}{d}V^2$

$\therefore \boxed{W_E = \frac{1}{2}CV^2}$ ** $\quad \because C = \frac{\epsilon A}{d}$.

$\because Q = CV \Rightarrow \boxed{W_E = \frac{1}{2}QV}$ **

$\because V = \frac{Q}{C} \Rightarrow W_E = \frac{1}{2}C\cdot\frac{Q^2}{C^2} \Rightarrow \boxed{W_E = \frac{1}{2}\frac{Q^2}{C}}$ **

→



$d_1 = d_2 = d/2$

→ If the capacitors are in series, then the total capacitance is

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

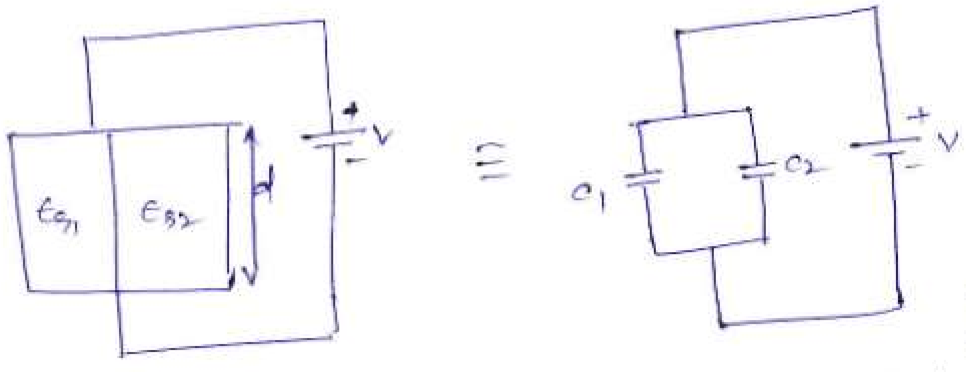
(or) $\boxed{C = \frac{C_1 C_2}{C_1 + C_2}}$

$C_1 = \frac{\epsilon A}{d/2} = \frac{2\epsilon_0\epsilon_{r1}A}{d}$

$C_2 = \frac{\epsilon A}{d/2} = \frac{2\epsilon_0\epsilon_{r2}A}{d}$

$\boxed{C = \frac{2\epsilon_0 A}{d(\epsilon_{r1}+\epsilon_{r2})}}$

→



$d_1 = d_2 = d$.

$A_1 = A_2 = A/2$

$C_1 = \frac{\epsilon A_1}{d} = \frac{\epsilon A}{2d}$

$C_2 = \frac{\epsilon A}{2d}$

If the capacitors are in parallel, then the total capacitance is

$\boxed{C = C_1 + C_2}$

$C = \frac{\epsilon_0\epsilon_{r1}A}{2d} + \frac{\epsilon_0\epsilon_{r2}A}{2d}$

$\boxed{C = \frac{\epsilon_0 A}{2d}\left[\epsilon_{r1} + \epsilon_{r2}\right]}$

ETR

Magneto - statics

→ The charges in motion i.e., flow of charges constitutes an Electric Current. Thus a current carrying conductor is always surrounded by a Magnetic field.

→ If the current flow is steady i.e., time invariant then the magnetic field produced is a steady magnetic field which is also a time Invariant.

→ The Direct Current (DC) is a steady flow of current hence magnetic field produced by a conductor carrying DC current is a static (or) steady Magnetic field.

→ The study of steady Magnetic field, existing in a given space produced due to the flow of Direct Current through a conductor is called "Magneto statics".

→ Generally Magnetic fields can be produced by

(a) permanent Magnet

(b) Direct Current

(c) Moving charges

(d) Time varying Electric field.

Magnetic field :—

→ If we place a magnetic material (or) a magnet in any medium, the region around a magnet in which the influence of the magnet can be experienced is called 'Magnetic field'. The existence of a field can be represented by an imaginary lines around the magnet which are called 'Magnetic Lines of force'.

→ The lines of force are called as "Magnetic Lines of Flux" or 'Magnetic flux lines'.

→ When a straight conductor carries a direct current, it produces a magnetic field around it all along its length. The lines of force in such a case are in the form of concentric circles in the planes at right angles to the conductor.

→ The direction of concentric circles around depends on the direction of current through a conductor.

→ A right hand Thumb rule is used to determine the direction of magnetic field around a conductor carrying a direct current.

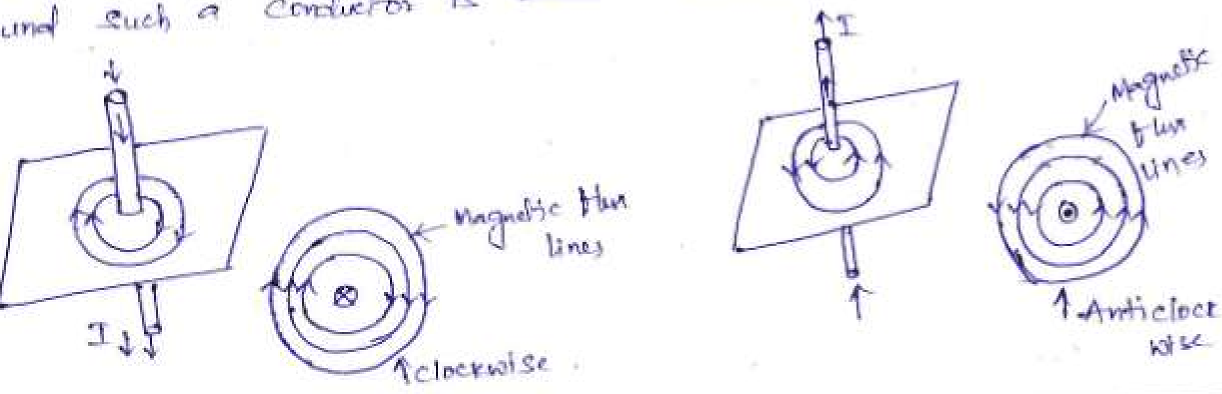
→ According to Right hand Thumb Rule, the direction of the current in current carrying conductor is along the direction of the thumb [i.e., the direction ~~of~~ in which the thumb pointing in that direction current flows in the conductor] and the direction of magnetic lines of flux around the conductor pointing in the direction of curled fingers.

→ Practically the current carrying conductor is represented by a small circle i.e., top view of straight conductor while the direction of current through it indicated by a 'cross' or a 'dot'.

→ The 'cross' indicates that the current direction is going into the plane of the paper, away from the observer.

→ The 'dot' indicates that the current direction is coming out of the plane of the paper coming towards the observer.

→ using right hand thumb rule, the direction of magnetic flux around such a conductor is either clockwise or Anti clockwise.



Magnetic field Intensity (H):—

→ The quantitative measure of strongness or weakness of the magnetic field is given by 'Magnetic field Intensity' or 'Magnetic field strength'.

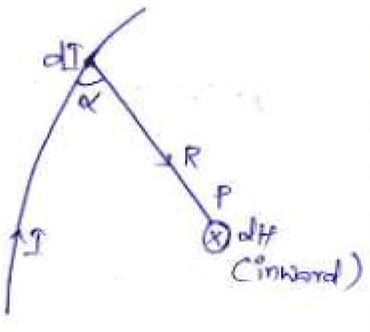
→ The magnetic field intensity at any point in the magnetic field is defined as the force experienced by a unit north pole of one weber strength, when placed at that point.

→ Magnetic flux lines are measured in Webers (Wb) while magnetic field intensity is measured in Newtons per weber (N/Wb) or Ampere per meter (A/m) or Ampere-turn/meter (AT/m).

→ It is denoted by $\bar{H}$.

Biot-Savart's Law:—

→ According to Biot-Savart's law, the differential magnetic field Intensity 'dH' produced at a point P (as shown in figure) by the differential current element 'Idl' is proportional to the product of Idl and the sine of the angle 'α' between the element and the line joining 'P' to the element and is inversely proportional to the square of the distance 'R' between 'P' and the element.



That is,

$$dH \propto \frac{I dl \sin\alpha}{R^2} \longrightarrow (1)$$

$$\Rightarrow \quad dH = \frac{K I dl \sin\alpha}{R^2} \longrightarrow (2)$$

where 'K' is the constant of proportionality. $K = 1/4\pi$

***

$$\therefore \boxed{dH = \frac{I dl \sin\alpha}{4\pi R^2}} \text{ Amp/mtr} \longrightarrow (3)$$

$\left\{ dH = \frac{\text{Amp-mtr}}{\text{mtr}^2} = A/m \right.$; $I \to$ Amp, $Idl \to$ Amp-mtr, $R \to$ mtr.

from the definition of cross product, eqn ③ is better put in vector form as

***

$$\boxed{dH = \frac{I dl \times a_R}{4\pi R^2} = \frac{I dl \times \bar{R}}{4\pi R^3}} \longrightarrow (4)$$

where $R = |\bar{R}|$ and $a_R = \frac{\bar{R}}{|\bar{R}|}$

$\left\{ \begin{aligned} dl \times a_R &= |dl||a_R| \sin\alpha \\ &= dl \cdot 1 \cdot \sin\alpha \end{aligned} \right.$

$\boxed{dl \times a_R = dl \sin\alpha}$

Thus the direction of dH can be determined by the Right hand Thumb rule.

Alternatively, we can use the right hand screw rule to determine the direction of dH; with the screw placed along the wire and pointed in the direction of current flow. The direction of advance of the screw is the direction of dH.

→ We can have different current distributions (like) such as line current, surface current and volume current.

→ If we define K as the surface current density in Ampere per meter and J as the volume current density in Ampere per meter square. The source elements are related as

$$\boxed{Idl = Kds = Jdv} \quad ***$$

Thus interms of the distributed current sources, the Biot-Savart's law becomes

$$\boxed{H = \int_L \frac{Idl \times a_R}{4\pi R^2}} \text{ (line current)} \longrightarrow (5)$$

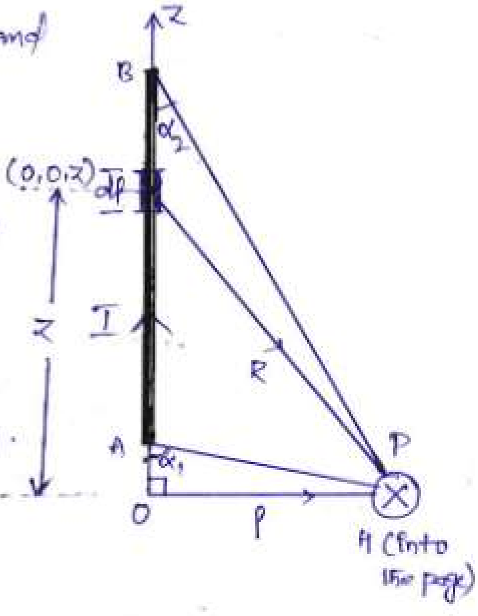
$$\boxed{H = \int_S \frac{Kds \times a_R}{4\pi R^2}} \text{ (Surface current)} \longrightarrow (6)$$

where a_R is a unit vector pointing from the differential element of current to the point of interest.

Field Intensity due to a finite Length of conductor :-

Assume a finite Length of a conductor AB and assume the conductor is placed along z-axis with its upper and lower ends respectively subtending angles α_2 and α_1 at P, the point at which H is to be measured (or determined). First formula to be derived will have to be applied accordingly.

Note that current flows from point A where $\alpha = \alpha_1$ to point B where $\alpha = \alpha_2$....



If we consider the contribution dH at P due to an element dl at (0,0,z) is

$$dH = \frac{Idl \times R}{4\pi R^3} \text{ A/m} \longrightarrow ①$$

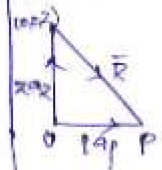
But $\bar{dl} = dz\,a_z$ and $\boxed{\bar{R} = \rho a_\rho - z a_z}$. So

$$d\bar{l} \times \bar{R} = dz\,a_z \times (\rho a_\rho - z a_z)$$

$$= dz\,\rho\, a_\phi - 0$$

$$\therefore d\bar{l} \times \bar{R} = \rho\, dz\, a_\phi \longrightarrow (2)$$

Since $z a_z + \bar{R} = \rho a_\rho$



$\Rightarrow \bar{R} = \rho a_\rho - z a_z$.

From ortho-Normality property

$a_\rho \times a_\phi = a_z \qquad a_\phi \times a_\rho = -a_z$

$a_\phi \times a_z = a_\rho \qquad a_z \times a_\phi = -a_\rho$

$a_z \times a_\rho = a_\phi \qquad a_\rho \times a_z = -a_\phi$

$a_\rho \times a_\rho = 0$

$a_\phi \times a_\phi = 0$

$a_z \times a_z = 0$.

Eqn (1) gives the Magnetic field Intensity due to a differential part of the conductor at particular point. To get total magnetic field Intensity due to a finite length of the conductor at that particular point, we need to integrate eqn (1) over the length of the conductor.

Hence
$$H = \int_A^B \frac{I \rho\, dz}{4\pi(\rho^2+z^2)^{3/2}}\, a_\phi \longrightarrow (3)$$

Let $z = \rho \cot\alpha$

$dz = -\rho \operatorname{cosec}^2\alpha\, d\alpha$

Limits:-

$z = A \Rightarrow \alpha = \alpha_1$

$z = B \Rightarrow \alpha = \alpha_2$

$z = \rho\cot\alpha$

$\rho^2 + z^2 = \rho^2 + \rho^2\cot^2\alpha$

$= \rho^2[1+\cot^2\alpha]$

$= \rho^2\operatorname{cosec}^2\alpha$

$\because \operatorname{cosec}^2\alpha - \cot^2\alpha = 1$

$\operatorname{cosec}^2\alpha = 1+\cot^2\alpha$

$$\therefore H = -\frac{1}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{I\rho^2 \operatorname{cosec}^2\alpha\, d\alpha}{4\pi\, \rho^3 \operatorname{cosec}^3\alpha}\, a_\phi$$

$$= -\frac{I}{4\pi\rho}\int_{\alpha_1}^{\alpha_2} \frac{1}{\operatorname{cosec}\alpha}\, d\alpha\, a_\phi = -\frac{I}{4\pi\rho}\int_{\alpha_1}^{\alpha_2} \sin\alpha\, d\alpha\, a_\phi$$

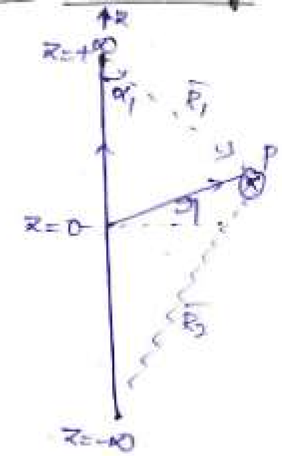
$$H = \frac{-I}{4\pi\rho}\left[-\cos\alpha\right]_{\alpha_1}^{\alpha_2} a_\phi$$

***

$$\therefore \boxed{H = \frac{I}{4\pi\rho}\left[\cos\alpha_2 - \cos\alpha_1\right] a_\phi}$$

This expression is applicable for any straight conductor of finite length. The conductor need not lie on the z-axis, but it must be straight. From the above eqn, we notice that $\bar{H}$ is always along the unit vector a_ϕ (i.e., along concentric circular paths).

Case (i) :— If the conductor has the length of '∞' [i.e., infinity] :—



If the length of the conductor is infinity and this conductor is symmetrically placed i.e., on either side of the z=0 plane the length of the conductor is same. we have to find magnetic field intensity due to this infinite length of conductor.

$$H = \int_{-\infty}^{\infty} \frac{I r \, dz}{4\pi R_1^3} a_\phi$$

$\boxed{\bar{R}_1 = -z a_z + r a_\rho \quad |\bar{R}_1| = \sqrt{r^2+z^2}}$

Here $\bar{R}_1 = -z a_z + r a_z$

$|\bar{R}_1| = \sqrt{r^2+z^2}$ $\qquad a_{R_1} = \dfrac{r a_\rho - z a_z}{\sqrt{r^2+z^2}}$

$$H = \int_{-\infty}^{\infty} \frac{I r \, dz}{4\pi [r^2+z^2]^{3/2}} a_\phi$$

Let $z = r \cot\alpha$

$dz = -r \operatorname{cosec}^2\alpha \, d\alpha$

Limits :—

when $z=-\infty$, $-\infty = \cot\alpha \Rightarrow \alpha = 180°$

when $z=\infty$, $\infty = \cot\alpha \Rightarrow \alpha = 0°$

$r^2+z^2 = r^2 + r^2\cot^2\alpha$

$= r^2 \operatorname{cosec}^2\alpha .$

$$H = -\int_{\pi}^{0} \frac{I r^2 \operatorname{cosec}^2\alpha \, d\alpha}{4\pi r^3 \operatorname{cosec}^3\alpha} a_\phi$$

$$= \frac{-I}{4\pi r} a_\phi \int_{\pi}^{0} \sin\alpha \, d\alpha = \frac{-I}{4\pi r} a_\phi \left[-\cos\alpha\right]_{\pi}^{0}$$

$$= \frac{I}{4\pi r} a_\phi \left[\cos 0 - \cos\pi\right] = \frac{I}{4\pi r} a_\phi [2]$$

$$\boxed{H = \frac{I}{2\pi r} a_\phi} \quad ***$$

This is the magnetic field intensity expression due to an infinite length of conductor.

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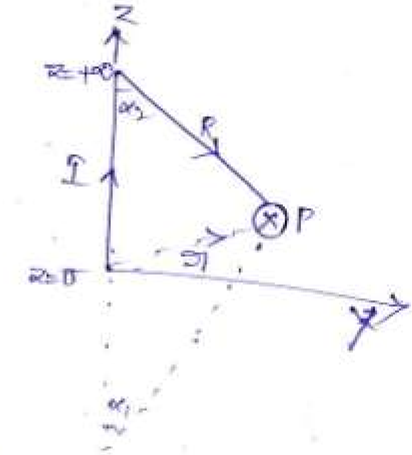
Case(ii):- Magnetic field Intensity Expression Due to Semi-infinite length of conductor :—



Since we have

$$H = \int \frac{I r\, dz}{4\pi R^3} a_\phi$$

for a Semi-infinite conductor

$$H = \int_0^\infty \frac{I r\, dz}{4\pi R^3} a_\phi \qquad \bar{R} = r a_\rho - z a_z$$

$$R = |\bar{R}| = \sqrt{r^2+z^2}$$

$$\therefore H = \int_0^\infty \frac{I r\, dz}{4\pi (r^2+z^2)^{3/2}} a_\phi$$

Let $z = r\cot\alpha$

$dz = -r\,\mathrm{cosec}^2\alpha\, d\alpha$

Limits:-

$z=0 \Rightarrow \cot\alpha = 0 \Rightarrow \alpha = \pi/2$

$z=\infty \Rightarrow \cot\alpha = \infty \Rightarrow \alpha = 0$

$$\therefore H = -\int_{\pi/2}^{0} \frac{I r^2 \mathrm{cosec}^2\alpha\, d\alpha}{4\pi r^3 \mathrm{cosec}^3\alpha} a_\phi$$

$$H = -\frac{I}{4\pi r}\int_{\pi/2}^{0} \sin\alpha\, d\alpha\, a_\phi = \frac{-I}{4\pi r}\left[-\cos\alpha\right]_{\pi/2}^{0} a_\phi$$

$$H = \frac{I}{4\pi r}\left[\cos\alpha\right]_{\pi/2}^{0} a_\phi = \frac{I}{4\pi r}\left[\cos 0 - \cos\pi/2\right] a_\phi = \frac{I}{4\pi r}[1-0]\, a_\phi$$

***

$$\therefore \boxed{H = \frac{I}{4\pi r} a_\phi}$$

Amperes Circuit Law :—

Since we have the formula for Magnetic field Intensity at a Particular point due to an infinite length of conductor of current I is at a distance of 'r' is

$$H = \frac{I}{2\pi r} a_\phi .$$

If we find the total magnetic field intensity due to an infinite length conductor at a distance 'r' from the conductor around the conductor. so by taking all points around the conductor with distance r_1 gives the circumference. so we have to find total field intensity due to conductor all points around the conductor with distance r_1 is

$$\boxed{\oint H \cdot dl = I_{enc}}$$ ***

$= H_{(radial)} \times (2\pi r_1)$ points

$= \frac{I}{2\pi r_1} \times 2\pi r_1 = I$

The above equation is called 'Maxwell's III eqn in Integral form'.

→ Ampere's law is easily applied to determine H. when the current distribution is symmetrical.

→ By applying stokes Theorem to the LHS of above eqn, we obtain

$$I_{enc} = \oint_L H \cdot dl = \int_S (\nabla \times H) \cdot ds \longrightarrow ①$$

But we have, $I_{enc} = \int_S J \cdot ds \longrightarrow ②$

By comparing eqns ① & ②, we have

*** $\boxed{\nabla \times H = J}$ - Amperes law in Differential (or) point form.

The above equation is called 'Maxwells III eqn in Differential form'.

→ we observe that $\nabla \times H = J (\neq 0)$, that is magneto static field is not conservative.

→ Statement:

Amperes circuit Law States that the Line Integral of H around a closed path is the same as the net current I_{enc} enclosed by the path.

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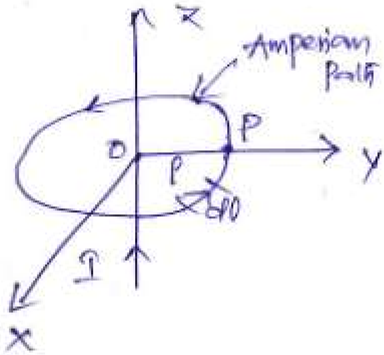
Applications of Ampere's Law :—

→ we apply Amperes Circuit law to determine H for some symmetrical Current distributions. we Consider an Infinite line Current, an infinite sheet Current and an Infinitely long coaxial transmission line. In each case, we apply $\oint_L H \cdot dl = I_{enc}$ for symmetrical Current distribution H is parallel or perpendicular to dl. when H is parallel to dl, $|H| =$ constant

Case (1) :— Infinite Line Current :—

consider an infinitely long filamentary current I along the z-axis as shown in figure.



To determine H at an observation point P, we allow a closed path to pass through P. This path, on which Ampere's law is to be applied, is known as an "Amperian path". we choose a concentric circle as the Amperian path. which shows that H is constant provided ρ is constant. Since this path enclosed the whole current I,

According to Ampere's law

$$I = \int H_\phi a_\phi \cdot \rho d\phi a_\phi = H_\phi \int \rho d\phi = H_\phi \cdot 2\pi\rho$$

(or) $$\boxed{H = \frac{I}{2\pi\rho} a_\phi}$$ ***

Note:- You can also follow the derivation of H for an infinite line.

Case(2) :— Infinite Sheet of Current :—

consider an Infinite sheet in the z=0 plane. If the sheet has a uniform current density $K = K_y a_y$ A/m as shown in figure.

By applying Ampere's law to the rectangular closed path 1-2-3-4-1 (Amperian path) gives

$$\boxed{\oint H \cdot dl = I_{enc} = K_y b} \quad *** \longrightarrow (1)$$

To evaluate the integral, first we need to have (devide) the infinite sheet as comprising of filaments; dH (field due to individual filament) above or below the sheet due to a pair of filamentary currents can be found.

As evident in figure, the resultant dH has only an x-component.

→ H on one side of the sheet is the negative of that on the other side.

→ owing to the infinite extent of the sheet, the sheet can be regarded as consisting of such filamentary pairs so that the characteristics of H for a pair are the same for the infinite current sheets, that is,

$$H = \begin{cases} H_0 a_x & , z > 0 \\ -H_0 a_x & , z < 0 \end{cases} \longrightarrow (2)$$

where H_0 is to be determined.

Evaluating the line integral of H in above eqn, along the closed path in figure gives

$$\oint H \cdot dl = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 \right) H \cdot dl$$

$$\oint H \cdot dl = 0(-a) + (-H_0)(-b) + 0(a) + H_0(b) = 2H_0 b \longrightarrow (3)$$

from eqns (1) & (3), we obtain $H_0 = \frac{1}{2} K_y$.

substituting H_0 in eqn (1), gives

$$H = \begin{cases} \frac{1}{2} K_y \, a_x & z > 0 \\ -\frac{1}{2} K_y \, a_x & z < 0 \end{cases} \longrightarrow (4)$$

In general, for an infinite sheet of current density K A/m,

$$\boxed{H = \frac{1}{2} K \times a_n} \; *** \longrightarrow (5)$$

where a_n is a unit normal vector directed from the current sheet to

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the point of interest.

Case(3):— Infinitely Long Coaxial Transmission Line :—

consider an infinitely long transmission line consisting of two concentric cylinders having their axes along the z-axis. The cross section of the line is shown in figure. where the z-axis is out of the page. The inner conductor has radius a and carries current I, while the outer conductor has inner radius b and thickness t and carries return current -I. we want to determine H everywhere. assuming that current is uniformly distributed in both conductors. Since the current distribution is symmetrical. we apply Ampere's law along the Amperian path for each of the four possible regions: $0 \le \rho \le a$, $a \le \rho \le b$, $b \le \rho \le b+t$ and $\rho \ge b+t$.

→ for region $0 \le \rho \le a$, we apply Ampere's law to path L_1, giving

$$\oint_L H \cdot dl = I_{enc} = \int J \cdot ds \longrightarrow ①$$

Since the current is uniformly distributed over the cross section

$$J = \frac{I}{\pi a^2} a_z , \quad ds = \rho \, d\phi \, d\rho \, a_z$$

$$I_{enc} = \int J \cdot ds = \frac{I}{\pi a^2} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{a} \rho \, d\phi \, d\rho = \frac{I}{\pi a^2} \pi \rho^2 = \frac{I\rho^2}{a^2}$$

Hence eqn① becomes

$$H_\phi \int_{L_1} dl = H_\phi \cdot 2\pi\rho = \frac{I\rho^2}{a^2}$$

(or) $$H_\phi = \frac{I\rho}{2\pi a^2} \longrightarrow ②$$

The region $a \le \rho \le b$, we use path L_2 as the Amperian path,

$$\oint_{L_2} H \cdot dl = I_{enc} = I$$

$$H_\phi \, 2\pi\rho = I \qquad \text{(or)} \qquad \boxed{H_\phi = \frac{I}{2\pi\rho}} \longrightarrow (3)$$

Since the whole current I is enclosed by L_2. Notice that eqn (3) is the independent of a. for region $b \le \rho \le b+t$, we use path L_3, getting

$$\oint_{L_3} H \cdot dl = H_\phi \cdot 2\pi\rho = I_{enc} \longrightarrow (4)$$

where $\qquad I_{enc} = I + \int J \cdot ds$

and J in this case is the current density (current per unit Area) of the outer conductor and is along $-a_z$, that is

$$J = \frac{I}{\pi[(b+t)^2 - b^2]} a_z$$

Thus

$$I_{enc} = I - \frac{I}{\pi[(b+t)^2 - b^2]} \int_{\phi=0}^{2\pi} \int_{\rho=b}^{\rho} \rho \, d\rho \, d\phi$$

$$= I \left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt} \right]$$

Substituting this in eqn (4), we have

$$H_\phi = \frac{I}{2\pi\rho} \left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt} \right] \longrightarrow (5)$$

for region $\rho \ge b+t$, we use path L_4, getting

$$\oint_{L_4} H \cdot dl = I - I = 0$$

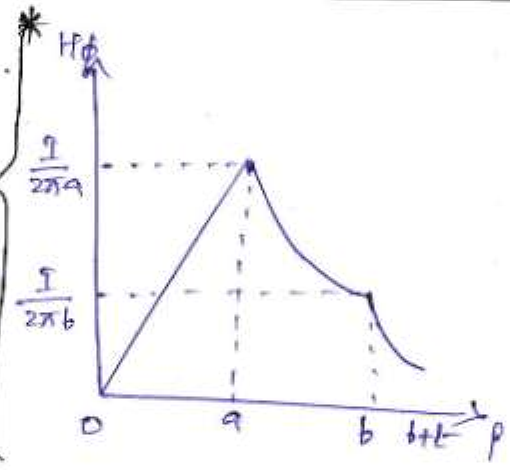
$$\text{(or)} \qquad H_\phi = 0 \longrightarrow (6)$$

Putting eqns (2) to (6) together gives

$$H = \begin{cases} \frac{I\rho}{2\pi a^2} a_\phi & 0 \le \rho \le a \\ I/2\pi\rho \; a_\phi & a \le \rho \le b \\ I/2\pi\rho \left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt} \right] a_\phi & b \le \rho \le b+t \\ & \rho \ge b+t \end{cases}$$

The magnitude of H is sketched in figure.

→ Ampere's Law can be used to find H only due to symmetrical current distribution for which it is possible to find a closed path over which 'H' is constant in magnitude.



Magnetic Flux Density :-

→ Magnetic flux Density is the number of flux lines passing through the particular Area.

→ The magnetic flux Density B is related to the magnetic field intensity 'H' according to

$$\boxed{B = \mu_0 H} \longrightarrow (1)$$

where $\mu \rightarrow$ permeability of a medium. Henry/mtr.
$\mu_0 \rightarrow$ permeability of free space, Henry/mtr
$\mu_r \rightarrow$ Relative permeability of a medium. (dimensionless)

$$\boxed{\mu_0 = 4\pi \times 10^{-7}\ H/m} \longrightarrow (2)$$

→ If we know the magnetic flux Density B. in a Surface Area 'S' then we can find the magnetic flux. i.e.,

$$\boxed{\psi = \int_S B \cdot ds} \longrightarrow (3)$$

where the magnetic flux 'ψ' is in webers (wb) and the magnetic flux Density is in Webers per square meter (wb/m²) or Teslas (T).

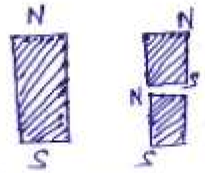
→ A magnetic flux line is a path to which 'B' is tangential at every point on the line. It is a line along which the needle of a magnetic compass will orient itself if placed in the presence of a magnetic field.

→ Each flux line is closed and has no beginning or end.

→ For a straight, current-carrying conductor the magnetic flux lines are closed and do not cross each other regardless of the current distribution.

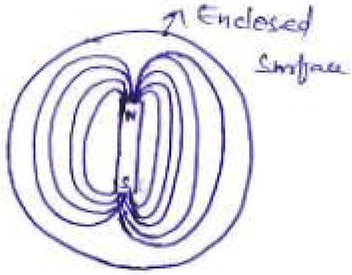
**→ Magnetic charges cannot be isolated.

For example, if we desire to have an isolated magnetic pole by dividing a magnetic bar successively into two, we end up with pieces each having North and South poles. We find it is impossible to separate the north pole from the south pole.



→ If we place a magnet in a closed surface, whatever the flux lines which are generated at North pole are entered at South pole. If we observe the no. of flux lines passes through an enclosed surface, those are zero. i.e.,

$$\boxed{\oint B\,ds = 0}$$ *** → "Maxwell's IV eqn in Integral form". ⟶ (4)



This equation is referred to as "Law of conservation of Magnetic Flux (or) Gauss's Law for Magnetostatic fields".

→ From the above equation, Magnetic flux lines are conservative even though the magnetostatic field is not conservative.

→ By applying the divergence theorem to eqn(4), we obtain

$$\oint_S B\cdot ds = \int_V \nabla\cdot B\,dv = 0$$

(or) $$\boxed{\nabla\cdot B = 0}$$ * ⟶ (5)

This is the "Maxwell's IV equation in Differential form".

→ From eqns (4) & (5), Magnetostatic fields have no sources or sinks.

→ Eqn(5) suggests that magnetic field lines are always continuous.

Magnetic scalar and Vector potentials :—

→ Magnetic potential could be scalar V_m or Vector A. To Define V_m and A involves recalling two important identities:

$$\boxed{\begin{array}{l} \nabla \times (\nabla V_m) = 0 \\ \nabla \cdot (\nabla \times A) = 0 \end{array}}$$

$\longrightarrow$ ① [Curl of Gradient of scalar = 0]

$\longrightarrow$ ② [Divergence of Curl of vector = 0]

which must always hold for any scalar field V_m and Vector field A.

→ we can define the magnetic scalar potential V_m (in amperes) as related to H according to

$$\boxed{H = -\nabla V_m} \quad \text{if } J = 0 \longrightarrow (3)$$

(or) $$\boxed{V_m = -\int H \, dl}$$

If this field magnetic potential is scalar, then it must satisfies eqn ①. otherwise it is a vector.

Since we have, $\nabla \times H = J \longrightarrow (4)$

from eqns ③ & ④, $\nabla \times (-\nabla V_m) = J$

$$-\nabla \times (\nabla V_m) = J \Rightarrow \boxed{\nabla \times (\nabla V_m) = -J}$$

for free space $J = 0$ i.e., from eqn(5)

$$\boxed{\nabla \times (\nabla V_m) = 0} \longrightarrow (5)$$

Then Assumed magnetic scalar potential satisfies eqn① then it is a scalar potential. [But it satisfies when $J=0$].

→ Magnetic scalar potential can exists in free space only.

→ V_m also satisfies Laplace's equation. hence

$$\boxed{\nabla^2 V_m = 0}. \quad (J=0).$$

→ Since we have (from Maxwell's IV Eqn).

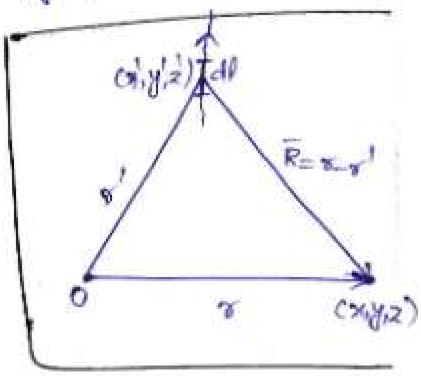
$$\boxed{\nabla \cdot B = 0} \longrightarrow (6)$$

To be a vector potential, it must satisfies eqn ②. so to satisfies eqn ②. We must ~~assume~~ define a vector potential A (Wb/m) such that,

$$\boxed{B = \nabla \times A} \longrightarrow (7)$$

Since we have $H = \int \frac{I\,dl \times \bar{R}}{4\pi R^3} \longrightarrow$ ⑧

where $\bar{R}$ is the distance vector from the line element dl at the source point (x', y', z') to the field point (x, y, z) as shown in fig. and $R = |\bar{R}|$, that is



$$R = |r - r'| = \left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{1/2} \longrightarrow ⑨$$

Hence

$$\nabla\left(\frac{1}{R}\right) = \frac{-\nabla \bar{R}}{R^2} \longrightarrow ⑩$$

$$\left\{ \begin{aligned} \nabla(U/V) &= \frac{V\nabla U - U\nabla V}{V^2} \\ \nabla(1/R) &= \frac{R\nabla 1 - 1\nabla \bar{R}}{R^2} \\ \nabla(1/R) &= \frac{-\nabla \bar{R}}{R^2} \end{aligned} \right.$$

But we have, $\nabla \bar{R} = \frac{\bar{R}}{|R|} = \frac{\bar{R}}{R}$

from Eqn (10),

$$\nabla(1/R) = \frac{-\bar{R}}{R^3} \qquad \left[\frac{-a_R}{R^2}\right]$$

$$\therefore \nabla(1/R) = \frac{-\left[(x-x')a_x + (y-y')a_y + (z-z')a_z\right]}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2\right]^{3/2}} = \frac{-\bar{R}}{R^3} \longrightarrow (11)$$

from Eqn's ⑧ & ⑪, $H = -\int \frac{I\,dl \times \nabla(1/R)}{4\pi}$

and

$$B = \frac{-\mu_0}{4\pi} \int_L I\,dl \times \nabla(1/R) \longrightarrow (12)$$

Since $B = \mu_0 H$ in free space

from vector Identity,

$$\nabla \times (fF) = f(\nabla \times F) + (\nabla f) \times F$$

$$-\nabla \times (fF) = -f(\nabla \times F) + (\nabla f) \times F$$

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$$-\nabla \times (fF) = -f(\nabla \times F) + F \times \nabla f$$

$$\Rightarrow F \times \nabla f = f(\nabla \times F) - \nabla \times (fF) \longrightarrow (13)$$

$f \to$ scalar field $\quad F \to$ vector field.

Let $f = 1/R$ and $F = dl$, we have from eqn (13),

$$dl \times \nabla(1/R) = \frac{1}{R}\nabla \times dl - \nabla \times (dl/R)$$

Since ∇ operates with respect to (x, y, z) while dl is a function of (x', y', z'), $\nabla \times dl = 0$. Hence

$$dl \times \nabla(1/R) = -\nabla \times \frac{dl}{R} \longrightarrow (14)$$

from eqns (12) and (14),

$$B = \frac{\mu_0}{4\pi}\int_L I(\nabla \times dl/R) = \nabla \times \int_L \frac{\mu_0 I\, dl}{4\pi R} \longrightarrow (15)$$

from eqn's (7) & (15), $\boxed{A = \int_L \frac{\mu_0 I\, dl}{4\pi R}}$ ***

→ Since we have $\psi = \int_S B \cdot ds$

But $B = \nabla \times A$, $\therefore \psi = \int_S (\nabla \times A) \cdot ds$

By applying stoke's Theorem for the above eqn.

$$\boxed{\psi = \oint_L A \cdot dl}$$ ***

from the above eqn, the magnetic flux through a given area can be found.

---

forces due to Magnetic fields :—

There are atleast three ways in which force due to magnetic fields can be experienced. The force can be (a) Due to a moving charged particle in a B field, (b) on a Current element in an external B field, or (c) between two Current elements.

(A) force on a charged particle :—

The electric force 'fe' on a stationary or moving electric charge 'Q' in an electric field is givenby Coulomb's experimental law and is related to the electric field intensity 'E' as

$$\boxed{f_e = QE} \longrightarrow (1)$$

This shows that if Q is positive, fe and E have the same direction.

→ A magnetic field can exert force only on moving charge. from experiments, it is found that the magnetic force 'fm' experienced by a charge 'Q' moving with a velocity 'v' in a magnetic field 'B' is

$$\boxed{f_m = Q\,v \times B} \longrightarrow (2)$$

This shows that fm is perpendicular to both 'v' and 'B'.

→ A comparison between the electric force 'fe' and the magnetic force 'fm' can be made. fe is independent of the velocity of the charge and can perform work on the charge and the change its kinetic energy. fm depends on the charge velocity and is normal to it. However fm cannot perform work because it is at right angles to the direction of motion of the charge ($f_m \cdot dl = 0$); it doesn't cause an increase in kinetic energy of the charge. The magnitude of fm is generally small in comparison to fe except at high velocities.

→ for a moving charge 'Q' in the presence of both electric and magnetic fields, the total force on the charge is given by

$$f = f_e + f_m$$

or

$$\boxed{F = Q(E + v \times B)} \longrightarrow (3)$$

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This is known as "Lorentz force Equation".

→ If the mass of the charged particle moving with in E and B fields is m, by Newton's second law of Motion.

$$\boxed{F = m\frac{dv}{dt} = Q(E + V \times B)} \longrightarrow (4)$$

The solution to this equation is important in determining the motion of charged particles in E and B fields.

(B) Force on a Current Element:—

To determine the force on a Current element Idl of a current carrying conductor due to the magnetic field B. We use convection current

$$\boxed{J = \rho_v v} \longrightarrow (1)$$

Since we have the relationship between Current elements:

$$\boxed{Idl = kds = Jdv} \longrightarrow (2)$$

combining eqns (1) & (2)

$$\boxed{Idl = \rho_v v\, dv = dQ v} \longrightarrow (3)$$

since $Q = \int \rho_v dv$

$dQ = \rho_v dv$

Alternatively, $Idl = \frac{dQ}{dt} dl = dQ\frac{dl}{dt} = dQv$

Hence

$$\boxed{Idl = dQv} \longrightarrow (4)$$

This shows that an elemental charge dQ moving with velocity v is equivalent to a conduction Current element Idl. Thus the force on a current element Idl in a magnetic field B is

$$\boxed{dF = Idl \times B} \longrightarrow (5)$$

If the Current I passes through a closed path L or circuit, the force on the circuit is given by

$$\boxed{F = \oint_L Idl \times B} \longrightarrow (6)$$

From the above eqns, the magnetic field produced by the current element Idl doesnot exert force on the element itself. The B field that exerts force on Idl must be due to another element.

→ If instead of the line current element Idl, we have surface current elements $K ds$ or a volume current element $J dv$, then

$$dF = K\,ds \times B \qquad \text{(or)} \qquad dF = J\,dv \times B$$

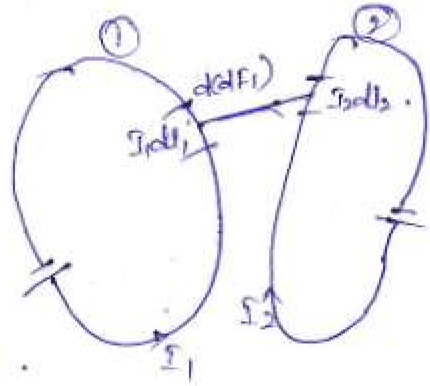
$$F = \int_S K\,ds \times B \qquad \text{or} \qquad F = \int_V J\,dv \times B$$

→ The magnetic field 'B' is defined as the force per unit current element.

(c) force between two current elements:—

consider the force between two elements $I_1 dl_1$ and $I_2 dl_2$. According to Biot-Savart's law, both current elements produce magnetic fields. So we may find the force $d(dF_1)$ on element $I_1 dl_1$ due to the field dB_2 produced by element $I_2 dl_2$ as shown in figure.

$$d(dF_1) = I_1 dl_1 \times dB_2 \longrightarrow (1)$$



where dF_1 → force on $I_1 dl_1$ due to the field produced in loop 2 due to the current I_2.

F_1 → force on loop 1 due to the field produced in loop 2 due to the current I_2.

But from Biot-Savart's law,

$$dB_2 = \frac{\mu_0 I_2 dl_2 \times a_{R21}}{4\pi R_{21}^2} \longrightarrow (2)$$

Hence

$$d(dF_1) = \frac{\mu_0 I_1 dl_1 \times (I_2 dl_2 \times a_{R21})}{4\pi R_{21}^2} \longrightarrow (3)$$

This equation is the law of force between two current elements, which expresses the force between two stationary charges. From eqn(3) we obtain the total force F_1 on current loop 1 due to current loop 2. That is

$$dF_1 = \int d(dF_1) = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_2} \frac{dl_1 \times (dl_2 \times a_{R21})}{R_{21}^2} \longrightarrow (4)$$

$$F_1 = \int dF_1 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1}\oint_{L_2} \frac{dl_1 \times (dl_2 \times a_{R21})}{R_{21}^2} \longrightarrow (5)$$

Eqn (5) gives the force F_1 on loop 1 due to the field produced in loop 2 due to the current I_2.

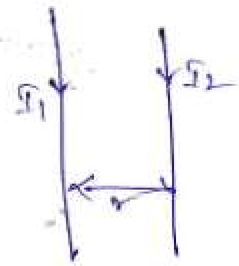
The force F_2 on loop 2 due to the magnetic field B_1 from loop 1 is obtained from by interchanging subscripts 1 and 2 in eqn (5).

It can be shown that $F_2 = -F_1$.

Ampere's force law :-

According to Ampere's law of force, the force F between two parallel wires carrying currents I_1 and I_2 is directly proportional to the individual currents and inversely proportional to the square of the distance between them i.e.,

$F \propto I_1 \quad F \propto I_2 \quad , F \propto 1/r^2$



$$F \propto \frac{I_1 I_2}{r^2}$$

$$F = \frac{K I_1 I_2}{r^2}$$ where $K \rightarrow$ proportionality constant

$$K = \frac{\mu}{4\pi}$$

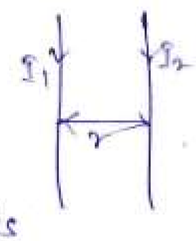
$$\therefore \boxed{F = \frac{\mu I_1 I_2}{4\pi r^2}} \; N$$

→ The force F is attractive if the two currents I_1 and I_2 are in the same direction and repulsive if in opposite directions.

Force between two long parallel Linear wires :-

→ Let us consider two long parallel wires of infinitely long, carrying currents I_1 and I_2 separated by a distance r.

The current I_2 produces a magnetic induction of B_2 at the portion of the current I_1. The force dF acting on an element dl_1 of this current is given by



$$dF = I_1 dl_1 \times B_2$$

$$dF = I_1 dl_1 B_2 \sin\theta \qquad \theta = 90^\circ$$

$$dF = |dF| = I_1 dl_1 B_2$$

Since we know that the expression for Magnetic field Intensity due to an infinite length of conductor at 'r' distance from the conductor is

$$B_2 = \frac{\mu I_2}{2\pi r}$$ [By taking magnitude, a_ϕ will cancel]

Since $H = \frac{I}{2\pi r} a_\phi$

$$\therefore \; dF = \frac{I_1 dl_1 \mu I_2}{2\pi r}$$

$$dF = \frac{\mu I_1 I_2 dl_1}{2\pi r}$$

***

$$\therefore \; \boxed{F = \frac{\mu I_1 I_2 l_1}{2\pi r}}$$

$$\frac{F}{l} = \frac{\mu I_1 I_2}{2\pi r}$$

This is the force per unit length. This eqn gives the definition of the Ampere.

Inductance:—

→ Inductor is a device to store the energy in the form of magnetic field, some times called as 'Inductance'.

→ If we twist a wire or a conductor of certain length into coil it becomes a basic Inductor.

Ex: — Coils, Solenoids, Toroidal Loop etc.

→ The magnetic flux lines are produced when the current passes through an Inductor. These flux lines passes through the conductor and links the current N-times.

→ When all the lines link all the turns, the total magnetic flux linkage 'λ' of the coil is equal to the total magnetic flux ϕ_m through the coil, times the no-of turns. i.e.,

$$\boxed{\text{flux Linkage} = \lambda = N\psi_m} \text{ wb-turn} \longrightarrow (1)$$

If the medium surrounding the circuit is linear, the flux linkage 'λ' is proportional to the current 'I' producing it. That is

$$\lambda \propto I$$

$$\Rightarrow \boxed{\lambda = LI} \longrightarrow (2)$$

where $L \rightarrow$ constant of proportionality

→ The Inductance 'L' is a property of the physical arrangement of the circuit. A circuit or part of a circuit that has inductance is called an Inductor.

from (1) & (2), $LI = N\psi_m$

$$\boxed{L = \frac{N\psi_m}{I}} \text{ H} \longrightarrow (3)$$

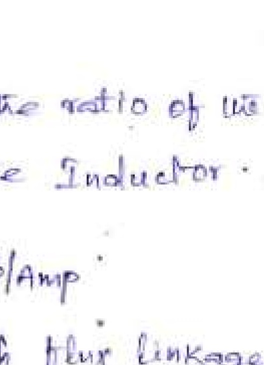
from eqn(3), we may define the Inductance as the ratio of the Magnetic flux linkage 'λ' to the current I through the Inductor.

→ Units for the Inductance are Henry, Wb/Amp.

→ Self Inductance is the inductance in which flux linkages are produced by the (inductor) itself only.
current passes in

→ Inductance may be regarded as a measure of, amount of magnetic energy is stored in an Inductor. The Magnetic Energy stored in an inductor is expressed as

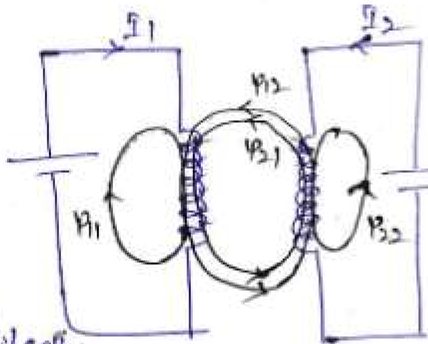
$$W_m = \frac{1}{2}LI^2 \text{ Joules.}$$



$$\Rightarrow \boxed{L = \frac{2W_m}{I^2}} \text{ Henrys}$$

→ Mutual Inductance is the Inductance in which flux linkages are produced due to the current passes through an another loop.

→ If we have two circuits carrying currents I_1 and I_2 as shown in figure, a magnetic interaction exists between the circuits. four component fluxes $\psi_{11}, \psi_{12}, \psi_{21}$ and ψ_{22} are produced. The



flux ψ_{11} represents the flux passing through (self flux) loop 1 due to current passes through loop 1 itself.

ψ_{22} represents the flux passing through the loop 2 due to the current passes through the loop 2 itself (self flux)

ψ_{12} → flux passing through Loop 1 due to the current passes through loop 2. (Mutual flux)

ψ_{21} → flux passing through Loop 2 due to the current passes through the loop 1. (Mutual flux).

→ If B_2 is the field produced in loop 2 due to the current flow through itself only. some of these flux lines are passes through the loop 1. If the Area of the loop 1 is S_1 then

$$\boxed{\psi_{12} = \int_{S_1} B_2 \cdot ds}$$

we defined, Mutual Inductance M_{12} as the ratio of the flux linkage $\lambda_{12} = N_1 \psi_{12}$ on circuit 1 to current I_2 : that is

$$\boxed{M_{12} = \frac{\lambda_{12}}{I_2} = \frac{N_1 \psi_{12}}{I_2}}$$

Similarly, M_{21} is the flux linkages of circuit 2 per unit current I_1 ; that is

$$\boxed{M_{21} = \frac{\lambda_{21}}{I_1} = \frac{N_2 \psi_{21}}{I_1}}$$

→ If the medium surrounding the circuits is Linear then

**
$$\boxed{M_{12} = M_{21}}$$

for a given Inductor, we find the self Inductance 'L' by taking these steps:

(1) choose a suitable coordinate system

(2) Let the Inductor carry current I.

(3) Determine B from Biot-Savart's law (or from Ampere's law if symmetry exists) and calculate ψ from $\psi = \int B \cdot ds$

(4) finally find L from $L = \frac{\lambda}{I} = \frac{N\psi}{I}$.

Coefficient of Coupling:- (K)

Since $M_{12} = \frac{N_1 \psi_{12}}{I_2} = \frac{N_1 (K_2 \psi_{22})}{I_2}$

& $M_{21} = \frac{N_2 \psi_{21}}{I_1} = \frac{N_2 (K_1 \psi_{11})}{I_1}$

for a Linear Medium, $M_{21} = M_{12}$

Hence $M^2 = M_{12} \cdot M_{21}$

$$= \frac{N_1 (K_2 \psi_{22})}{I_2} \cdot \frac{N_2 (K_1 \psi_{11})}{I_1}$$

$$M^2 = K_1 K_2 \left(\frac{N_1 \psi_{11}}{I_1}\right) \cdot \left(\frac{N_2 \psi_{22}}{I_2}\right)$$

$$M^2 = K_1 K_2 L_1 L_2 \Rightarrow M = \sqrt{K_1 K_2 L_1 L_2}$$

Let $\cancel{\sqrt{K_1 K_2}}$ $K_1 K_2 = K^2 \Rightarrow$

$$\therefore M = K\sqrt{L_1 L_2}$$

$$\boxed{K = \frac{M}{\sqrt{L_1 L_2}}}$$

$L_1 \rightarrow$ Inductance of coil 1. $L_2 \rightarrow$ Inductance of coil 2

$M \rightarrow$ Mutual Inductance

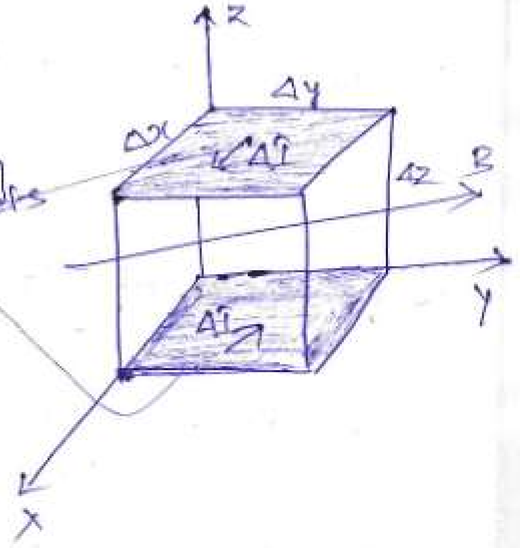
→ Series Aiding: $L_{eq} = (L_1 + L_2 + 2M)$ H

Series opposing: $L_{eq} = L_1 + L_2 - 2M$ H

Parallel Aiding: $L_{eq} = \frac{L_1 + L_2 - M^2}{L_1 + L_2 - 2M}$

Parallel opposing: $L_{eq} = \frac{L_1 + L_2 - M^2}{L_1 + L_2 + 2M}$

Magnetic Energy :-



Consider a differential volume in a magnetic field. Let the volume be covered with conducting sheets at the top and bottom surfaces with current ΔI. We assume that the whole region is filled with such differential volumes. Each volume has an Inductance

$$\Delta L = \frac{\Delta \psi}{\Delta I} = \frac{\mu H \, \Delta x \Delta z}{\Delta I}$$

where $\Delta I = H \Delta y$.

Since we have, $\Delta W_m = \frac{1}{2} \Delta L \, \Delta I^2$

$$\Delta W_m = \frac{1}{2} \mu H^2 \Delta x \Delta y \Delta z \Rightarrow \Delta W_m = \frac{1}{2} \mu H^2 \Delta v$$

The magnetostatic energy density w_m (in J/m^3) is defined as

$$w_m = \lim_{\Delta v \to 0} \frac{\Delta W_m}{\Delta v} = \frac{1}{2} \mu H^2$$

Hence,

$$\boxed{w_m = \frac{1}{2} \mu H^2 = \frac{1}{2} B \cdot H = \frac{B^2}{2\mu}}$$

Thus the energy in a magnetostatic field in a linear medium is

*** $$W_m = \int w_m \, dv$$

$$\boxed{W_m = \frac{1}{2} \int B \cdot H \, dv = \frac{1}{2} \int \mu H^2 \, dv}$$

Maxwell's Equations (Time varying fields)

Introduction:

→ In static Electromagnetic fields, Electric and Magnetic fields are independent of each other, whereas in dynamic EM fields, the two fields are interdependent i.e., a time varying electric field necessarily involves a corresponding time varying magnetic field.

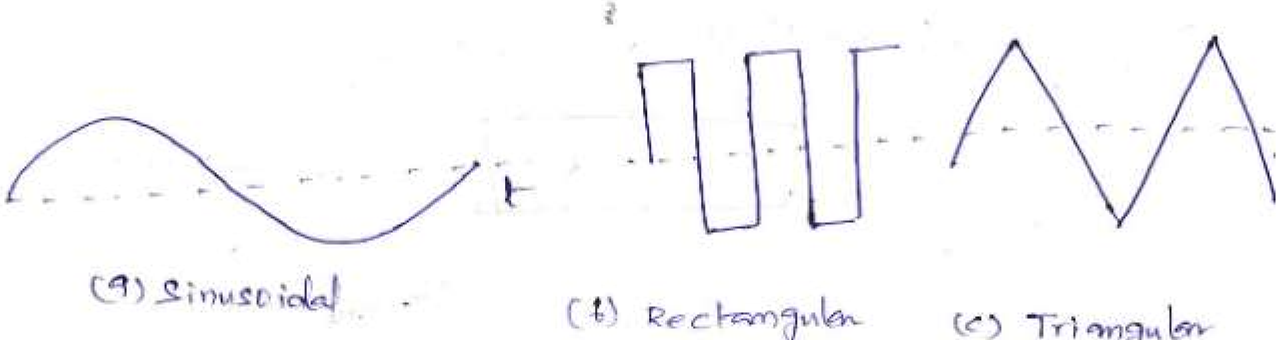
→ The time varying EM fields, represented by E(x,y,z,t) and H(x,y,z,t) are of more practical value than static EM fields.

→ The Electrostatic fields are usually produced by static electric charges, whereas magnetostatic fields are due to motion of electric charges with uniform velocity (Direct current) or static magnetic charges (Magnetic poles); time-varying fields or waves are usually due to accelerated charges or time varying currents.

→ Any pulsating current will produce radiation (time-varying fields).

→ stationary charge causes Electrostatic fields
steady currents causes Magnetostatic fields
Time-varying currents causes Electromagnetic waves (or fields)

→ Examples of time-varying currents are



(a) Sinusoidal (b) Rectangular (c) Triangular

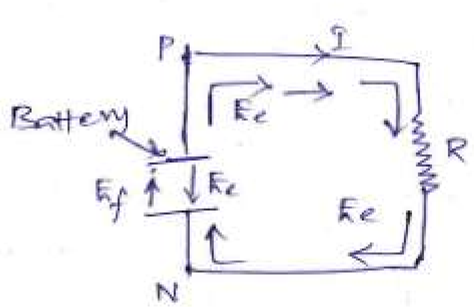
faraday's law:—

According to faraday's, a static magnetic field produces no current flow. but a time varying field produces an induced voltage (called electromotive force or simply emf) in a closed circuit. which causes a flow of current.

→ The induced emf, V_{emf} (in volts) in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit. This is called "faraday's law". and it can be expressed as

$$\boxed{V_{emf} = \frac{-d\lambda}{dt} = -N\frac{d\psi}{dt}} \longrightarrow (1)$$

where $\lambda = N\psi$ is the flux linkage, 'N' is the number of turns in the circuit; and ψ is the flux through each turn. The negative sign shows that the induced voltage acts in such a way as to oppose the flux producing it. This is called ~~Lentz~~ "Lenz's law" and it emphasizes that the direction of current flow in the circuit is such that the induced magnetic field produced by the induced current will oppose the change in original magnetic field.



consider a circuit of figure shown, where the battery is a source of emf. The electrochemical action of the battery results in an emf-produced field E_f. Due to the accumulation of charge at the battery terminals, an electrostatic field E_e ($= -\nabla V$) also exists.

∴ The total electric field at any point is

$$\boxed{E = E_f + E_e} \longrightarrow (2)$$

→ E_f is zero outside the battery, E_f and E_e have opposite directions in the battery and the direction of E_e inside the battery is opposite to that outside it.

If we integrate eqn(2) over the closed circuit, we have.

$$\boxed{\oint_L E \cdot dl = \oint_L E_f \cdot dl + 0 = \int_N^P E_f \cdot dl} \quad \text{(Through Battery)} \longrightarrow (3)$$

where $\oint E_e \cdot dl = 0$ because E_e is conservative.

The emf of the battery is the line integral of the emf-produced field. that is,

$$V_{emf} = \int_N^P E_f \cdot dl = -\int_N^P E_e \cdot dl = IR$$

Since E_f and E_e are equal but opposite within the battery.

→ It is important to note the following facts:

(1) An electrostatic field E_e cannot maintain a steady current in a closed circuit since

$$\boxed{\oint_L E_e \cdot dl = 0 = IR}$$

(2) An emf-produced field E_f is nonconservative.

(3) Except in electrostatics, voltage and potential difference are usually not equivalent.

Transformer Electromotive forces (EMF) :-

Since we have,

$$\boxed{V_{emf} = -N\frac{d\varphi}{dt}} \longrightarrow (1)$$

for a circuit with a single turn (N=1), the above eqn becomes

$$V_{emf} = -\frac{d\varphi}{dt} \longrightarrow (2)$$

interms of E and B, eqn(2) can be written as

$$V_{emf} = \oint_L E \cdot dl = -\frac{d}{dt}\int_S B \cdot ds \longrightarrow (3)$$

where 'φ' has been replaced by $\int_S B \cdot ds$ and 'S' is the Surface Area of the circuit bounded by the closed path 'L'. from the above eqn(3), both electric and magnetic fields are present and are interrelated. The variation of flux with time as in eqn(1) or eqn(3) may be caused in three ways:

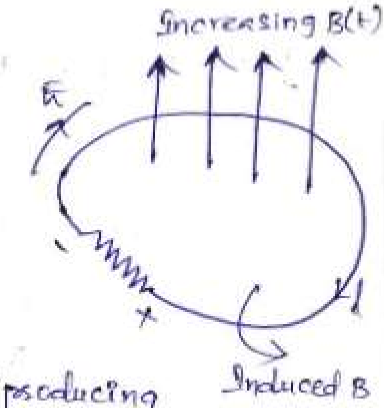
(1) By having a stationary loop in a time varying B field

(2) By having a time varying loop area in a static B field

(3) By having a time varying loop area in a time varying B field.

(1) Stationary Loop in Time Varying B field (Transformer EMF):-

In following figure, a stationary conducting loop is in time varying magnetic B field. Equation (3) becomes



$$\boxed{V_{emf} = \oint_L E \cdot dl = -\int_S \frac{\partial B}{\partial t} \cdot ds} \longrightarrow (4)$$

This emf induced by the time-varying current (producing the time varying B field) in a stationary loop is often referred to as 'transformer emf';

By applying stokes theorem to the middle term in eqn(4), we obtain

$$\boxed{\int_S (\nabla \times E) \cdot ds = -\int_S \frac{\partial B}{\partial t} \cdot ds} \longrightarrow (5)$$

for the two integrals to be equal, their integrands must be equal; that is

$$\boxed{\nabla \times E = -\frac{\partial B}{\partial t}} \longrightarrow (6).$$

This is one of the Maxwell's Equations for time-varying fields. It shows that the time varying 'E' field is not conservative ($\nabla \times E \neq 0$).

(2) Moving Loop in static B field (Motional EMF):—

When a conducting loop is moving in a static B field, an emf is induced in the loop. Since we know the force on a charge moving with uniform velocity 'v' in a magnetic field B is

$$\boxed{f_m = Q\, v \times B} \longrightarrow (7)$$

we define the motional electric field 'E_m' as

$$\boxed{E_m = \frac{f_m}{Q} = v \times B} \longrightarrow (8)$$

If we consider a conducting loop, moving with uniform velocity v & consisting of a large number of free electrons, the emf induced in the loop is

$$\boxed{V_{emf} = \oint_L E_m \cdot dl = \oint_L (v \times B) \cdot dl} \longrightarrow (9)$$

This type of emf is called 'Motional emf' or 'Flux Cutting emf' because it is due to motional action.

Ex:- Motors, Generators and Alternators etc.

and we know that.

$$f_m = I\, l \times B \longrightarrow (10)$$

(or)

$$f_m = I l B \longrightarrow (11)$$

and eqn(9) becomes

$$V_{emf} = vBl \longrightarrow (12)$$

By applying stokes Theorem to eqn (9), we have

$$\int_S (\nabla \times E_m) \cdot ds = \int_S \nabla \times (v \times B) \cdot ds$$

$$\boxed{\nabla \times E_m = \nabla \times (v \times B)} \longrightarrow (13)$$

(3) Moving Loop in Time-Varying field :-

A moving conducting loop is in a time-varying magnetic field. Both transformer emf and motional emf are present. Combining eqn's (4) and (9) gives the total emf as

$$\boxed{V_{emf} = \oint_L E \cdot dl = -\int_S \frac{\partial B}{\partial t} \cdot ds + \oint_L (v \times B) \cdot dl} \longrightarrow (14)$$

or from eqns (6) and (13),

$$\boxed{\nabla \times E = -\frac{\partial B}{\partial t} + \nabla \times (v \times B)} \longrightarrow (15)$$

Inconsistency of Amperes law:-

We know the point form of Amperes circuital law as it applies to steady magnetic fields,

$$\nabla \times H = J \longrightarrow (1)$$

and show it's inadequacy for time varying conditions by taking the divergence on each side,

$$\nabla \cdot \nabla \times H = \nabla \cdot J \longrightarrow (2)$$

Since the divergence of the curl is identically zero [i.e $\nabla \cdot \nabla \times H = 0$].

from eqn(2), $\nabla \cdot J = 0 \longrightarrow (3)$

But from the continuity equation, $\nabla \cdot J = -\frac{\partial \rho_v}{\partial t}$

But $\frac{\partial \rho_v}{\partial t} \neq 0$ so $\nabla \cdot J$ must not be zero.

$\therefore$ This inconsistency in Amperes law can be overcome by adding one more to the Maxwell's Equation.

Displacement Current Density:

The inconsistancy in Amperes law overcome by adding one more term to Maxwell's Equation

$$\boxed{\nabla \times H = J + J_d} \longrightarrow (1)$$

where J_d is to be determined and defined.

The divergence of the curl of any vector is zero. Hence

$$\nabla \cdot (\nabla \times H) = 0 = \nabla \cdot J + \nabla \cdot J_d \longrightarrow (2)$$

Since we have the Equation for continuity is

$$\nabla \cdot J = -\frac{\partial \rho_v}{\partial t} \longrightarrow (3)$$

To satisfies eqn(2), $\nabla \cdot J + \nabla \cdot J_d = 0$

$$\nabla \cdot J_d = -\nabla \cdot J = \frac{\partial \rho_v}{\partial t}$$

$$\nabla \cdot J_d = \frac{\partial (\nabla \cdot D)}{\partial t} = \nabla \cdot \frac{\partial D}{\partial t}$$

(or)

$$\boxed{J_d = \frac{\partial D}{\partial t}} \longrightarrow (4)$$

from (1) & (4) $\boxed{\nabla \times H = J + \frac{\partial D}{\partial t}}$ **

This is the Maxwell's Equation (based on Amperes circuit law) for a time varying field. The term $J = \partial D/\partial t$ is known as 'Displacement current density' and J is the 'conduction current density' ($J = \sigma E$).

→ without the term J_d, the propagation of Electromagnetic waves (e.g. radio or TV waves) would be impossible.

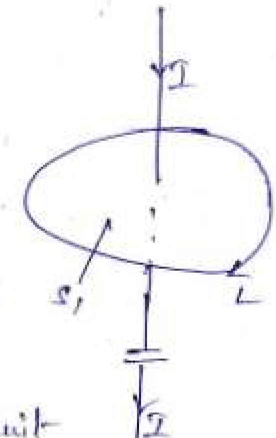
→ At low frequencies, J_d is usually neglected compared with J.

→ At Radio frequencies, the two terms are comparable.

→ Based on the Displacement Current Density, we define the displacement current as

$$I_d = \int J_d \cdot ds = \int \frac{\partial D}{\partial t} \cdot ds$$

The displacement current is a result of time-varying electric field.



A typical example of such current is a result of time varying electric field. A typical example of such current is the current through a capacitor when an alternating voltage source is applied to its plates.

Applying an unmodified form of Ampere's circuit law to a closed path 'L' shown in figure gives

$$\boxed{\oint_L H \cdot dl = \int_{S_1} J \cdot ds = \oint I_{enc} = I}$$

where 'I' current through the conductor and 'S_1' is the flat surface bounded by L.

Maxwell's Equations and word statements:—

→ The electromagnetic equations are known as 'Maxwell's Equations'. Since Maxwell con Each differential equation has its integral counterpart: one form may be derived from the other with the help of stoke's theorem (or) the divergence theorem. The equations are as follows, with the dot superscript indicating partial derivates with respect to time.

	Differential form	Integral form
(i)	$\nabla \times H = J + J_d$	$\oint H \cdot dl = \int (J + J_d) \cdot ds$
(ii)	$\nabla \times E = -\frac{\partial B}{\partial t}$	$\oint E \cdot dl = -\int \frac{\partial B}{\partial t} \cdot ds$
(iii)	$\nabla \cdot D = \rho_v$	$\oint D \cdot ds = \int \rho_v \, dv = Q$
(iv)	$\nabla \cdot B = 0$	$\oint B \cdot ds = 0$

statements of respective Maxwell's equations as follows:

(i) The magnetomotive force around a closed path is equal to the conduction current plus the time derivative of the electric displacement through any surface bounded by the path.

(ii) The electromotive force around a closed path is equal to the time derivative of the magnetic displacement through any surface bounded by the path.

(iii) The total electric displacement through the surface enclosing a volume is equal to the total charge within the volume.

(iv) The net magnetic flux emerging through any closed surface is zero.

→ The time derivative of electric displacement is called 'Displacement current'. The term electric current is then generalized in meaning to include both conduction currents and displacement currents.

→ If the time derivative of magnetic displacement can be considered as being a magnetic current.

The first two Maxwell equations can also be stated as:

(i) The magnetic voltage around a closed path is equal to the electric current through the path.

(ii) The electric voltage around a closed path is equal to the magnetic current through the path.

Conditions at a Boundary Surface :-

Maxwell's Equations in the differential form express the relationship that must exist between the four field vectors E, D, H and B at any point within a continuous medium. In this form, because they involve space derivatives, they cannot be expected to yield information at points of discontinuity in the medium. However, the integral forms can always be used to determine what happens at the boundary surface between different media.

- The following statements can be made regarding the electric and magnetic fields at any surface of discontinuity.

(a) The tangential component of 'E' is continuous at the surface. That is, it is the same just outside the surface as it is just inside the surface.

(b) The tangential component of 'H' is continuous across a surface except at the surface of a perfect conductor. At the surface of a perfect conductor the tangential component of 'H' is discontinuous by an amount equal to the surface current per unit width.

(c) The normal component of 'B' is continuous at the surface of discontinuity.

(d) The normal component of 'D' is continuous if there is no surface charge density. Otherwise D is discontinuous by an amount equal to the surface charge density.

The proof of these boundary conditions is obtained by a direct application of Maxwell's equations at the boundary between the media.

UNIT-IV ⑥

→ If the field exists in a region consisting of two different media, the conditions that the field must satisfy at the interface separating the media are called "Boundary conditions". These conditions are helpful in determining the field on one side of the boundary if the field on the other side is known.

We shall consider the boundary conditions at an interface separating

→ Dielectric - Dielectric

→ Dielectric - conductor.

To determine the boundary conditions, we need to use Maxwell's Equations:

$$\boxed{\oint E\cdot dl = 0} \longrightarrow (1)$$

and

$$\boxed{\oint_S D\cdot ds = Q_{enc}} \longrightarrow (2)$$

where Q_{enc} is the free charge enclosed by the surface 'S'. Also we need to decompose the electric field intensity 'E' into two orthogonal components

$$\boxed{E = E_t + E_n} \longrightarrow (3)$$

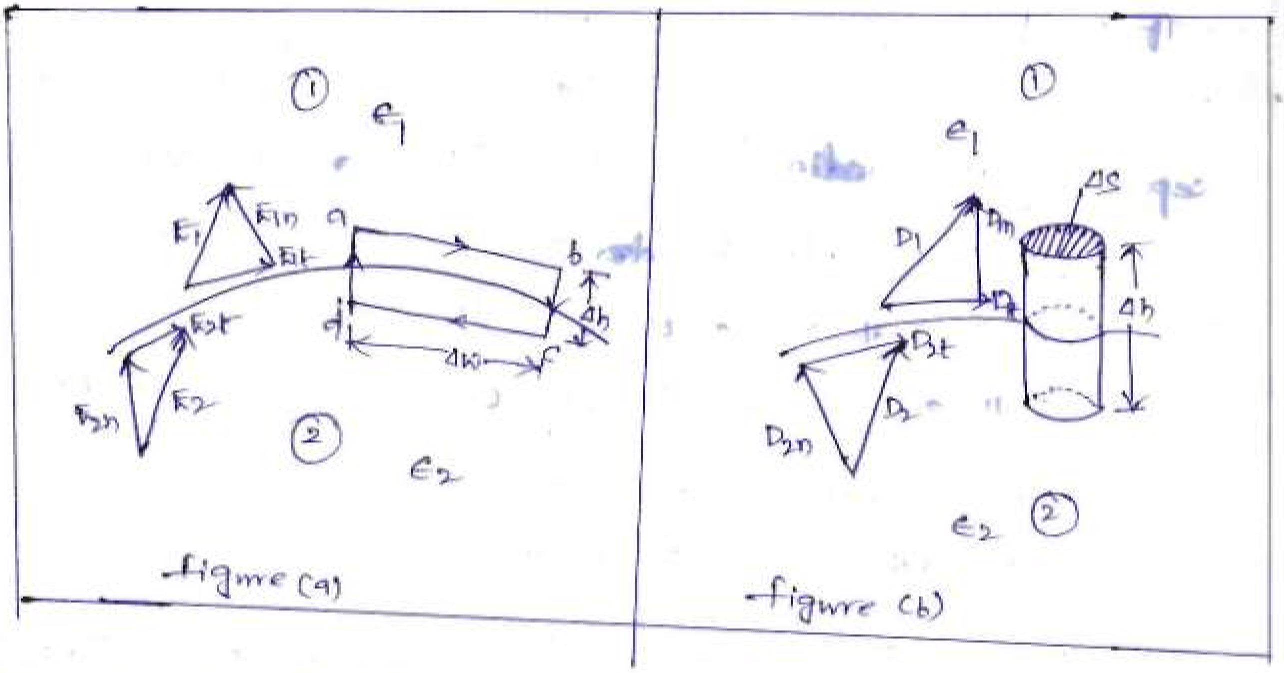
where E_t and E_n are respectively, the tangential and normal components of 'E' to the interface of interest. A similar decomposition can be done for the electric flux density 'D'.

(i) Dielectric - Dielectric Boundary conditions :—

Consider the E field existing in a region that consists of two different dielectrics characterized by $\epsilon_1 = \epsilon_0\epsilon_{r1}$ and $\epsilon_2 = \epsilon_0\epsilon_{r2}$ as shown in figure below. The fields E_1 and E_2 in media 1 and 2, respectively, can be decomposed as

$$E_1 = E_{1t} + E_{1n} \longrightarrow (4)$$

$$E_2 = E_{2t} + E_{2n} \longrightarrow (5)$$



We apply Eqn's (1) & (2) to the closed path abcda of figure (a). Assuming that the path is very small with respect to the spatial variation of E. We obtain

$$0 = E_{1t}\Delta W - \cancel{E_{1n}\frac{\Delta h}{2}} - E_{2n}\frac{\Delta h}{2} - E_{2t}\Delta W + \cancel{E_{2n}\frac{\Delta h}{2}} + \cancel{E_{1n}\frac{\Delta h}{2}} \longrightarrow (3)$$

where $E_t = |E_t|$ and $E_n = |E_n|$.

Eqn (3) becomes, $E_{1t}\Delta W - E_{2t}\Delta W = 0$

$$(E_{1t} - E_{2t})\Delta W = 0$$

As $\Delta h \to 0$,

$$\Rightarrow \boxed{E_{1t} = E_{2t}} \;*** \longrightarrow (4)$$

Thus the tangential components of E are the same on the two sides of the boundary. In other words, E_t undergoes no change on the boundary and it is said to be continuous across the boundary. Since $D = \epsilon E = D_t + D_n$. Eqn (4) can be written as

$$\frac{D_{1t}}{\epsilon_1} = E_{1t} = E_{2t} = \frac{D_{2t}}{\epsilon_2}$$

(or)

$$\boxed{\frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}} \;*** \longrightarrow (8)$$

That is, D_t undergoes some change across the interface. Hence D_t is said to be discontinuous across the interface.

Similarly, if we apply eqn(2) to the pillbox (cylindrical Gaussian surface) of figure (b). The contribution due to the sides vanishes. Allowing $\Delta h \to 0$ gives

$$\Delta Q = \rho_s \Delta S = D_{1n} \Delta S - D_{2n} \Delta S$$

(or)

$$\boxed{D_{1n} - D_{2n} = \rho_s} \longrightarrow (9)$$

where $\rho_s \to$ free charge density placed deliberately at the boundary.

If no free charges exist at the interface (i.e., charges are not deliberately placed there), $\rho_s = 0$ then the eqn(9) becomes

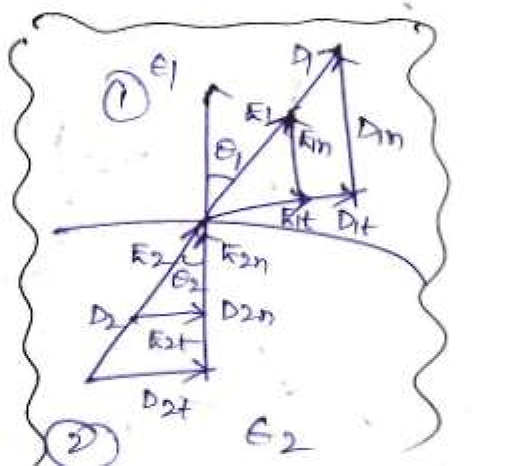
$$\boxed{D_{1n} = D_{2n}} \longrightarrow (10)$$

Thus the normal component of D is continuous across the interface, that is, D_n undergoes no change at the boundary. Since $D = \epsilon E$, eqn (10) can be written as

$$\boxed{\epsilon_1 E_{1n} = \epsilon_2 E_{2n}} \longrightarrow (11)$$

showing that the normal component of E is discontinuous at the boundary. Eqns (8), (9) and (10) are collectively referred to as "Boundary conditions"; they must be satisfied by an electric field at the boundary separating two different dielectrics.

→ The boundary conditions are usually applied in finding the electric field on one side of the boundary given the field on the other side. Besides this, we can use the boundary conditions to determine the "refraction of the electric field across the interface.



consider D_1 or E_1 and D_2 or E_2 making angles θ_1 and θ_2 with the normal to the interface as illustrated in figure. Using eqn(7), we have

$$E_1 \sin\theta_1 = E_{1t} = E_{2t} = E_2 \sin\theta_2$$

or

$$\boxed{E_1 \sin\theta_1 = E_2 \sin\theta_2} \longrightarrow (12)$$

similarly by applying eqn (10) or (11), we get

$$\epsilon_1 E_1 \cos\theta_1 = D_{1n} = D_{2n} = \epsilon_2 E_2 \cos\theta_2$$

(or)

$$\boxed{\epsilon_1 E_1 \cos\theta_1 = \epsilon_2 E_2 \cos\theta_2} \longrightarrow (13)$$

Dividing eqn (12) by eqn (13) gives

$$\boxed{\frac{\tan\theta_1}{\epsilon_1} = \frac{\tan\theta_2}{\epsilon_2}} \longrightarrow (14)$$

since $\epsilon_1 = \epsilon_0 \epsilon_{r1}$ and $\epsilon_2 = \epsilon_0 \epsilon_{r2}$, eqn (14) becomes

**

$$\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_{r1}}{\epsilon_{r2}}} \longrightarrow (15)$$

This is the "law of refraction" of the electric field at a boundary free of charge (since $\rho_s = 0$ is assumed at the interface). Thus, an interface between two dielectrics produces bending of the flux lines as a result of unequal polarization charges that accumulate on the opposite sides of the interface.

(ii) Dielectric - conductor Boundary conditions :—

following figure shows the case of conductor - Dielectric Boundary conditions. The conductor is assumed to be perfect (i.e. $\sigma \to \infty$ or $\rho_c \to 0$). Although such a conductor is not realizable for most practical purposes, we may regard conductors such as copper and silver as though they were perfect conductors.

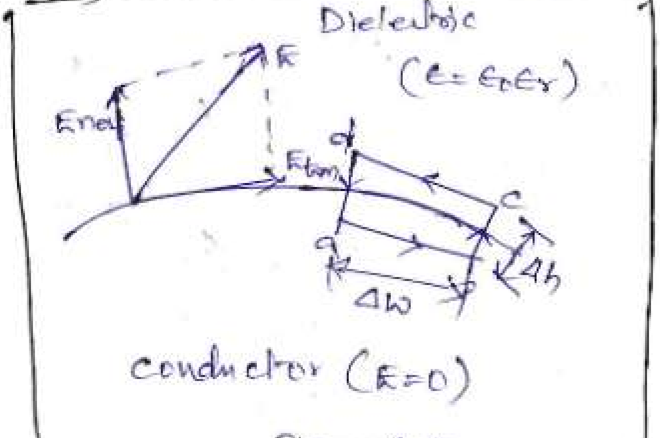


figure (a)

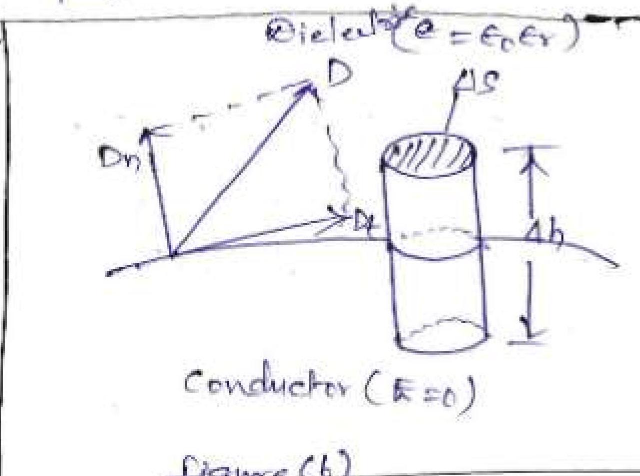


figure (b)

EMTL
UNIT-IV (8)

To determine the boundary conditions for a dielectric-conductor interface, we need to use Maxwell's equations. They are

$$\boxed{\oint E \cdot dl = 0} \longrightarrow (1)$$

and

$$\boxed{\oint D \cdot ds = Q_{enc}} \longrightarrow (2)$$

where $Q_{enc} \rightarrow$ free charge enclosed by the surface 's'.

Applying eqn(1) to the closed path abcda of figure (a) gives

$$0 = 0 \cdot \Delta w + 0 \cdot \frac{\Delta h}{2} + E_n \cdot \frac{\Delta h}{2} - E_t \cdot \Delta w - E_n \cdot \frac{\Delta h}{2} - 0 \cdot \frac{\Delta h}{2} \longrightarrow (3)$$

As $\Delta h \rightarrow 0$,

$$\boxed{E_t = 0} ** \longrightarrow (4)$$

Similarly, by applying eqn(2) to the cylindrical pill box of figure (b) and letting $\Delta h \rightarrow 0$ we get

$$\boxed{\Delta Q = D_n \cdot \Delta S - 0 \cdot \Delta S} \longrightarrow (5)$$

because $D = \epsilon E = 0$ inside the conductor. Eqn (5) may be written as

$$D_n = \frac{\Delta Q}{\Delta S} = \rho_s$$

(or) $\boxed{D_n = \rho_s} \longrightarrow (6)$

Thus under static conditions, the following conclusions can be made about a perfect conductor:

(1) No electric field may exist within a conductor; that is, ~~considering our conclusion~~

$$\boxed{\rho_v = 0, \quad E = 0} \longrightarrow (7)$$

(2) Since $E = -\nabla V = 0$, there can be no potential difference between any two points in the conductor; that is, a conductor is an equipotential body.

(3) An electric field E must be external to the conductor and must be normal to its surface; that is

$$\boxed{D_t = \epsilon_0 \epsilon_r E_t = 0 \quad , \quad D_n = \epsilon_0 \epsilon_r E_n = \rho_s} \longrightarrow (8)$$

An important application of the fact that $E=0$ inside a conductor is in "Electrostatic screening or shielding".

Electro-magnetic Wave Characteristics - I

→ Wave:— Wave is defined as a type of energy.

→ If the wave has both Electric & Magnetic field components then that wave is called 'Electromagnetic wave'.

→ These Electromagnetic waves are used to carry the information from one place (Source) to (destination) another place.

Ex:— FM, AM etc

→ The transmission of energy from one place to another place by the electromagnetic waves called 'Electromagnetic wave propagation".

→ In this, we study the characteristics of Electromagnetic waves while propagating through the different materials and different mediums [or Media].

→ To study the characteristics of the EM waves in different Media & materials, we need to solve the Maxwell's Equations.

→ In this, we study the behavior of EM waves in different Materials & Mediums which are

* free space (or Lossless or Non conducting medium)
* Lossless Dielectrics [$\sigma = 0$, $\varepsilon = \varepsilon_r \varepsilon_0$, $\mu = \mu_0 \mu_r$ (or) $\frac{\sigma}{\omega\varepsilon} << 1$]
* Lossy Dielectrics [$\sigma \neq 0$, $\varepsilon = \varepsilon_0 \varepsilon_r$, $\mu = \mu_0 \mu_r$]
* Good Conductors [$\sigma \cong \infty$, $\varepsilon = \varepsilon_0$, $\mu = \mu_0 \mu_r$ (or) $\frac{\sigma}{\omega\varepsilon} >> 1$]

→ A free space is a space, which doesn't interfere with any Electric & Magnetic field components & Gravitational fields also. So for free space σ(conductivity) $= 0$, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$.

→ Permittivity (ε):—

is defined as the storing ability of electric field.

→ permeability (μ):—

is defined as the storing capacity of the Magnetic field.

$$\mu = \mu_0 \mu_r$$

where $\mu \to$ permeability of the Medium

$\mu_0 \to$ permeability of free space

$\mu_0 = 4\pi \times 10^{-7}$ H/m

$\mu_r \to$ relative permeability.

→ permittivity can be

$$\varepsilon = \varepsilon_0 \varepsilon_r$$

where $\varepsilon \to$ permittivity of the Medium

$\varepsilon_0 \to$ permittivity of free space

$\varepsilon_r \to$ Relative permittivity.

$$\varepsilon_0 = 8.854 \times 10^{-12} = \frac{1}{36\pi \times 10^{9}} \text{ farad/Mts}$$

→ Homogeneous Medium:—

is a Medium in which the Quantities σ, μ, ε are constant throught the Medium.

→ Isotropic Medium:—

in which the Directions of D [Electric Flux Density] and E [Electric field Intensity] are same when ε is scalar.

General Wave Equation :-

→ In general, the wave equations can be obtained by relating the space and time variations of Electric & Magnetic fields using Maxwell's Equations.

→ To obtain the General wave equation, let us assume that electric and Magnetic fields exists in linear, homogeneous and isotropic medium with the parameters σ, μ and ϵ.

→ The Maxwells Equations are given by

$$① \leftarrow \nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} = -\mu\frac{\partial\bar{H}}{\partial t}$$

$$② \leftarrow \nabla\times\bar{H} = J_c + J_D = \sigma\bar{E} + \frac{\partial\bar{D}}{\partial t} = \sigma\bar{E} + \epsilon\frac{\partial\bar{E}}{\partial t}$$

$$③ \leftarrow \nabla\cdot\bar{B} = 0 \quad \text{i.e.,} \quad \nabla\cdot\bar{H} = 0$$

$$④ \leftarrow \nabla\cdot\bar{D} = \rho_v \quad \text{i.e.,} \quad \nabla\cdot\bar{E} = 0 \quad (\text{for free space } \rho_v = 0)$$

→ Wave Equation for Electric field ($\bar{E}$):

To eliminate $\bar{H}$ from eqn ①, take Curl on both the sides of eqn ①

$$\nabla\times(\nabla\times\bar{E}) = \nabla\times\left(-\mu\frac{\partial\bar{H}}{\partial t}\right) = -\mu\left(\nabla\times\frac{\partial\bar{H}}{\partial t}\right) \longrightarrow (5)$$

∇ → represents the differentiation (variation) with respect to space.

$\partial/\partial t$ → represents the differentiation (variation) with respect to time.

Both are independent to each other, the operators can be interchanged

$$\nabla\times(\nabla\times\bar{E}) = -\mu\frac{\partial}{\partial t}(\nabla\times\bar{H}) \longrightarrow ⑥$$

from eqn ②, $\nabla\times\bar{H} = \sigma\bar{E} + \epsilon\frac{\partial\bar{E}}{\partial t}$

$$\nabla\times\nabla\times\bar{E} = -\mu\frac{\partial}{\partial t}\left[\sigma\bar{E} + \epsilon\frac{\partial\bar{E}}{\partial t}\right] = -\mu\sigma\frac{\partial\bar{E}}{\partial t} - \mu\epsilon\frac{\partial^2\bar{E}}{\partial t^2} \longrightarrow ⑦$$

Now, from the vector identity

$$\nabla\times\nabla\times\bar{E} = \nabla(\nabla\cdot\bar{E}) - \nabla^2\bar{E} \longrightarrow ⑧$$

Since $\nabla\cdot\bar{E} = 0$. so eqn ⑧ becomes

$$\nabla \times \nabla \times \bar{E} = -\nabla^2 \bar{E} \longrightarrow (9)$$

from eqns (7) & (9),

$$\cancel{-}\nabla^2 \bar{E} = \cancel{-}\sigma \mu \frac{\partial \bar{E}}{\partial t} \cancel{-} \mu \varepsilon \frac{\partial^2 \bar{E}}{\partial t^2}$$

$$\therefore \boxed{\nabla^2 \bar{E} = \sigma \mu \frac{\partial \bar{E}}{\partial t} + \mu \varepsilon \frac{\partial^2 \bar{E}}{\partial t^2}} \longrightarrow (10)$$

Eqn (10) called 'Instantaneous vector wave eqn' or 'Helmholtz eqn'.

→ multiplying the eqn (10) on bothsides with ε, then

$$\nabla^2 \varepsilon \bar{E} = \mu \sigma \frac{\partial \varepsilon \bar{E}}{\partial t} + \mu \varepsilon \frac{\partial^2 \varepsilon \bar{E}}{\partial t^2}$$

$$\Rightarrow \boxed{\nabla^2 \bar{D} = \mu \sigma \frac{\partial \bar{D}}{\partial t} + \mu \varepsilon \frac{\partial^2 \bar{D}}{\partial t^2}} \longrightarrow (11)$$

This is the wave eqn in $\bar{D}$ for uniform field.

→ Wave eqn for Magnetic field :—

To eliminate $\bar{E}$ from eqn (2), take Curl on both the sides of (2),

$$\nabla \times \nabla \times \bar{H} = \nabla \times (\bar{J_c} + \bar{J_d}) = \nabla \times \left(\sigma \bar{E} + \varepsilon \frac{\partial \bar{E}}{\partial t}\right)$$

$$\nabla \times \nabla \times \bar{H} = \sigma (\nabla \times \bar{E}) + \varepsilon \left[\nabla \times \frac{\partial \bar{E}}{\partial t}\right]$$

∇ & $\frac{\partial}{\partial t}$ are independent to each other, so the operator can be interchanged,

$$\nabla \times \nabla \times \bar{H} = \sigma (\nabla \times \bar{E}) + \varepsilon \frac{\partial}{\partial t}(\nabla \times \bar{E}) \longrightarrow (12)$$

from eqn (1), eqn (12) can be written as

$$\nabla \times \nabla \times \bar{H} = \sigma \left(-\mu \frac{\partial \bar{H}}{\partial t}\right) + \varepsilon \frac{\partial}{\partial t}\left(-\mu \frac{\partial \bar{H}}{\partial t}\right)$$

$$\boxed{\nabla \times \nabla \times \bar{H} = -\mu \sigma \frac{\partial \bar{H}}{\partial t} - \mu \varepsilon \frac{\partial^2 \bar{H}}{\partial t^2}} \longrightarrow (13)$$

from vector identity, $\nabla \times \nabla \times \bar{H} = \nabla(\nabla \cdot \bar{H}) - \nabla^2 \bar{H}$

$$\nabla \times \nabla \times \bar{H} = -\nabla^2 H \quad [\text{since } \nabla \cdot \bar{H} = 0] \longrightarrow (14)$$

from eqns (13) & (14), $\boxed{\nabla^2 H = \sigma \mu \frac{\partial \bar{H}}{\partial t} + \mu \varepsilon \frac{\partial^2 \bar{H}}{\partial t^2}} \longrightarrow (14)$

Eqn(14) is called 'Instantaneous Vector Wave Eqn in H' or "Helmholtz Equation".

→ Multiplying the eqn(14) with 'μ' on both the sides,

$$\nabla^2(\mu\bar{H}) = \mu\sigma\frac{\partial(\mu\bar{H})}{\partial t} + \mu\varepsilon\frac{\partial^2(\mu\bar{H})}{\partial t^2}$$

$$\boxed{\nabla^2\bar{B} = \mu\sigma\frac{\partial\bar{B}}{\partial t} + \mu\varepsilon\frac{\partial^2\bar{B}}{\partial t^2}} \longrightarrow (15)$$

This is the wave eqn for uniform field in $\bar{B}$.

→ Eqns (10) & (14) are the wave Equations for any conducting medium.

→ The standard partial differential wave equation frequently encountered in Mechanical Engineering is of the form,

$$\nabla^2 x = \frac{1}{v^2}\frac{\partial^2 x}{\partial t^2} \longrightarrow (16)$$

where 'x' is any desired field vector or component and 'v' is the velocity of the wave.

By comparing eqns (10), (14) & (16),

→ for free space, $\sigma = 0$, then eqns (10) & (14) becomes

$\varepsilon = \varepsilon_0,\ \mu = \mu_0$

$$\nabla^2\bar{E} = \cancel{-\omega^2\mu_0\varepsilon_0\bar{E}}\ \mu_0\varepsilon_0\frac{\partial^2\bar{E}}{\partial t^2} \longrightarrow (17) \longleftarrow -$$

$$\nabla^2\bar{H} = \cancel{-\omega^2}\ \mu_0\varepsilon_0\frac{\partial^2\bar{H}}{\partial t^2} \longrightarrow (18)$$

By comparing eqns (16), (17) & (18), we get

$$v = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$$

Where $\mu_0 = 4\pi\times10^{-7}$ H/m, $\varepsilon_0 = \frac{1}{36\pi\times10^9}$ F/m

so $\boxed{v = 3\times10^8 \text{ m/sec}}$

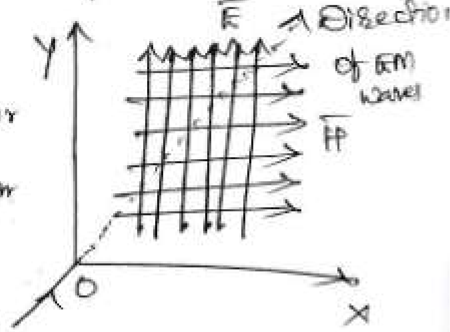
This is the velocity of light and hence when referred to EM or radio wave it is denoted by C- the velocity of light.

Thus $v = C = 3\times10^8$ m/sec.

** so in free space, Electromagnetic waves travels with the speed of light.

Plane wave & uniform plane wave in free or Empty space :-

→ Any wave which is from far distance is looking as a plane wave.
Ex:- ocean.



(1) At any point in space, the electric field vector $\bar{E}$ & Magnetic field vector $\bar{H}$ are perpendicular to each other and perpendicular to the direction of propagation.

→ A uniform plane wave in which $\bar{E}$ & $\bar{H}$ lie in a plane and have the same value everywhere in that plane at any fixed instant.

for z= Constant say defines a Surface.

In a uniform plane wave space variations of $\bar{E}$ & $\bar{H}$ are zero over z= Constant plane. This implies the fields have neither 'x' nor 'y' dependen

i.e., $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$ for all field Components.

(2) velocity of propagation of wave in free space is given by

$$C = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3\times10^8 \text{ m/sec}$$

and in free space, charge density (ρ) & Current density (σ) are zero.

(3) whatever may be the frequency, the electromagnetic waves travels in space with the velocity of light.

Solution of Maxwell's eqn for uniform plane wave :—

→ If the phase is same for all points on a plane surface, it is called 'plane wave' and if amplitude also constant over the plane surface it is known as 'uniform plane wave'.

→ for uniform plane wave, only one component is present and rest two are zero.

→ Let us now find the solution of Maxwells equation in Electric field intensity E, which is propagating in z-direction, then $\frac{\partial E}{\partial x}=0$ & $\frac{\partial E}{\partial y}=0$ and further since the wave is propagating in z-direction so neither the electric field nor the magnetic fields are there along z-direction so $E_z=0$, $H_z=0$.

Thus $\nabla^2 E = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$

$$a_x\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_x + a_y\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_y + a_z\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_z = \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2}\left[E_x a_x + E_y \cdot a_y + E_z \cdot a_z\right]$$

Since $E_z=0$, $\frac{\partial E_x}{\partial x}=\frac{\partial E_y}{\partial y}=\frac{\partial E_x}{\partial y}=\frac{\partial E_y}{\partial x}=0$ & $\frac{\partial^2 E}{\partial x^2}=\frac{\partial^2 E}{\partial y^2}=0$.

$$\therefore a_x\left(\frac{\partial^2 E_x}{\partial z^2}\right)+a_y\left(\frac{\partial^2 E_y}{\partial z^2}\right)=\mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\left[E_x a_x + E_y a_y\right]$$

By comparing the similar components on both the sides,

$$\boxed{\begin{aligned}\frac{\partial^2 E_x}{\partial z^2} &= \mu_0\varepsilon_0\frac{\partial^2 E_x}{\partial t^2}\\ \frac{\partial^2 E_y}{\partial z^2} &= \mu_0\varepsilon_0\frac{\partial^2 E_y}{\partial t^2}\end{aligned}}$$

Similarly for H-fields:

$$\boxed{\begin{aligned}\frac{\partial^2 H_x}{\partial z^2} &= \mu_0\varepsilon_0\frac{\partial^2 H_x}{\partial t^2}\\ \frac{\partial^2 H_y}{\partial z^2} &= \mu_0\varepsilon_0\frac{\partial^2 H_y}{\partial t^2}\end{aligned}}$$

Take anyone among the above eqns and find the solution for that,

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} \longrightarrow (1)$$

Let for any time harmonic function,

$$E_x = E_0 e^{j\omega t}$$

$$\frac{\partial E_x}{\partial t} = j\omega E_0 e^{j\omega t} = j\omega E_x$$

$$\frac{\partial^2 E_x}{\partial t^2} = j\omega (j\omega) E_0 e^{j\omega t} = -\omega^2 E_x \longrightarrow (2)$$

from (1) & (2),

$$\frac{\partial^2 E_x}{\partial z^2} = -\omega^2 \mu_0 \varepsilon_0 E_x \Rightarrow \left[\frac{\partial^2}{\partial z^2} + \omega^2 \mu_0 \varepsilon_0\right] E_x = 0 \longrightarrow (3)$$

Let $\frac{\partial}{\partial z} = D$

so $(D^2 + \omega^2 \mu_0 \varepsilon_0) E_x = 0$

for finding the Auxilliary solution,

$$D^2 + \omega^2 \mu_0 \varepsilon_0 = 0 \Rightarrow D^2 = j^2 \omega^2 \mu_0 \varepsilon_0$$

$$\boxed{D = \pm j\omega \sqrt{\mu_0 \varepsilon_0}} \longrightarrow (4)$$

so the solution for eqn (3) is

$$\boxed{E_x = K_1 e^{-j\omega\sqrt{\mu_0 \varepsilon_0}\, z} + K_2 e^{+j\omega\sqrt{\mu_0 \varepsilon_0}\, z}}$$

Since $D = \alpha \pm j\beta \longrightarrow (5)$

from eqns (4) & (5), $\alpha = 0$, so there is no attenuation.

$\beta = \omega\sqrt{\mu_0 \varepsilon_0} \rightarrow$ phase shift constant (or) phase constant

β is measured in rad/m

i.e., $E_x = K_1 e^{-j\beta z} + K_2 e^{+j\beta z} \longrightarrow (6)$

Let K_1 & K_2 are the constants with respect to 'z' but are functions of 't'.

Let us assume K_1 and K_2 as

$$K_1 = E_0^+ e^{+j\omega t} \quad \text{and} \quad K_2 = E_0^- e^{-j\omega t}$$

so
$$\boxed{E_x = E_0^+ e^{j(\omega t - \beta z)} + E_0^- e^{j(\omega t + \beta z)}} \rightarrow (7)$$

To find the electric field in time domain, take the real part of above eqn. then we get

$$E_x = Re\left[E_0^+ e^{j(\omega t - \beta z)} + E_0^- e^{+j(\omega t + \beta z)}\right]$$

$$\Rightarrow \boxed{E_x = E_0^+ \cos(\omega t - \beta z) + E_0^- \cos(\omega t + \beta z)} \ V/m$$

Transverse Nature of Uniform plane wave :—

→ To show that the uniform plane wares transverse in nature, we have to take the wave equation.

→ from the wave equation due to Electric field, in which the electric field is not dependent on 'x' and 'y' components but is a function of 'z' and 't' only. Such a wave is called 'Uniform plane wave' and is a special case of EM wave propagation.

→ The Maxwell wave eqn in 'E' is given by

$$\nabla^2 E = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} \longrightarrow (1)$$

$$a_x\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)E_x + a_y\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]E_y + a_z\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]E_z = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}\left[E_x a_x + E_y a_y + E_z a_z\right]$$

Let the wave be propagating in z-direction, so for a uniform plane wave

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0 \Rightarrow \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial y^2} = 0 \quad \& \quad E_z = H_z = 0$$

$$\therefore \frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}, \quad \frac{\partial^2 E_y}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

for free space, charge density is zero so from Maxwells eqn

$$\nabla \cdot D = 0 \Rightarrow \varepsilon \nabla \cdot E = 0 \Rightarrow \nabla \cdot E = 0.$$

so $\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \cdot \longrightarrow (2)$

for uniform plane wave, 'E' is independent of 'x' & 'y', so

$$\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} = 0$$

from (2) $\Rightarrow \frac{\partial E_z}{\partial z} = 0 \cdot \Rightarrow \frac{\partial^2 E_z}{\partial z^2} = 0 \cdot$

So There is no 'E_z' Component in the z-direction.

from (1), $\frac{\partial^2 E_z}{\partial t^2} = 0 \cdot$

This implies that 'E_z' must either be

(i) zero (or) (ii) constant in time (or) (iii) Increasing uniformly with time.

→ If E_z = constant & $E_z = kt$, It will not be a part of wave motion.

Therefore $E_z = 0$, Similarly $H_z = 0$.

→ This means that the Component of E- & H- fields of a uniform-plane wave in the direction of propagation are zero.

Relation between E & H for uniform plane wave :—

→ for finding the relation between 'E and H', we have to solve the Maxwell's eqn

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\begin{vmatrix} a_x & a_y & a_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} = -\frac{\partial}{\partial t}\left[a_x \cdot B_x + a_y B_y + a_z B_z\right]$$

for a uniform plane wave progressing in z-direction has no x- and y- components i.e., $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$ and no E_z & H_z components.

i.e., $E_z = 0$ & $H_z = 0$.

⑥

So $\begin{vmatrix} a_x & a_y & a_z \\ 0 & 0 & \partial/\partial z \\ E_x & E_y & 0 \end{vmatrix} = -\mu \frac{\partial}{\partial t}\left[H_x a_x + H_y a_y + H_z a_z\right]$

By expanding,

$$a_x\left[\frac{-\partial H_y}{\partial z}\right] - a_y\left[\frac{-\partial E_x}{\partial z}\right] = -\mu\frac{\partial}{\partial t}\left[H_x a_x + H_y a_y\right]$$

By equating the similar components

$$\boxed{\frac{\partial E_y}{\partial z} = \mu\frac{\partial H_x}{\partial t}} \longrightarrow ① \qquad \boxed{\frac{\partial E_x}{\partial z} = -\mu\frac{\partial H_y}{\partial t}} \longrightarrow ②$$

$$\nabla \times H = \frac{\partial D}{\partial t} = \epsilon\frac{\partial E}{\partial t}$$

$$\begin{vmatrix} a_x & a_y & a_z \\ 0 & 0 & \partial/\partial z \\ H_x & H_y & 0 \end{vmatrix} = \epsilon\frac{\partial}{\partial t}\left[E_x a_x + E_y a_y + E_z a_z\right]$$

By expanding, $a_x\left[\frac{-\partial H_y}{\partial z}\right] - a_y\left(\frac{-\partial H_x}{\partial z}\right) = \epsilon\frac{\partial E_x}{\partial t}a_x + \epsilon\frac{\partial E_y}{\partial t}a_y$

By equating the similar components

$$\boxed{\frac{\partial H_y}{\partial z} = -\epsilon\frac{\partial E_x}{\partial t}} \longrightarrow ③ \qquad \boxed{\frac{\partial H_x}{\partial z} = \epsilon\frac{\partial E_y}{\partial t}} \longrightarrow ④$$

from eqn ③,

$$\frac{\partial H_y}{\partial z} = -\epsilon\frac{\partial E_x}{\partial t}$$

since $E_x = E_0^+ e^{j(\omega t - \beta z)}$

$$\frac{\partial E_x}{\partial t} = j\omega E_0^+ e^{j(\omega t - \beta z)}$$

In order to get 'H_y' component, we need to integrate eqn ③ with respect to 'z'. So by doing that

$$\int\frac{\partial H_y}{\partial z}.\partial z = -\int\epsilon\frac{\partial E_x}{\partial t}.\partial z = -\int\epsilon \cdot j\omega E_0^+ e^{j(\omega t - \beta z)}\partial z$$

$$H_y = -j\omega\epsilon\left[\frac{E_0^+ e^{j(\omega t - \beta z)}}{-j\beta}\right] = \frac{\omega\epsilon}{\beta}E_x$$

$$\frac{H_y}{E_x} = \frac{\omega\epsilon}{\beta}$$

Since $\beta = \omega\sqrt{\mu\epsilon}$

$$\therefore \frac{H_y}{E_x} = \frac{\omega\epsilon}{\omega\sqrt{\mu\epsilon}} = \sqrt{\epsilon/\mu}$$

$$\therefore \boxed{\frac{E_x}{H_y} = \sqrt{\frac{\mu}{\epsilon}}} \longrightarrow (5)$$

from eqn (4), $\frac{\partial E_y}{\partial t} = j\omega E_0^+ e^{j(\omega t - \beta z)}$

$$\int \frac{\partial H_x}{\partial z} \cdot \partial z = \int \epsilon \frac{\partial E_y}{\partial t} \cdot \partial z = \epsilon \int j\omega E_0^+ e^{j(\omega t - \beta z)} \partial z$$

$$H_x = \frac{j\omega\epsilon E_0^+ e^{j(\omega t - \beta z)}}{-j\beta} = \frac{-\omega\epsilon}{\beta} E_y$$

$$\therefore \boxed{\frac{H_x}{E_y} = \frac{-\omega\epsilon}{\beta}}$$

Since $\beta = \omega\sqrt{\mu\epsilon}$

$$\therefore \boxed{\frac{H_x}{E_y} = -\sqrt{\epsilon/\mu}} \longrightarrow (6)$$

But $E = |E| = \sqrt{E_x^2 + E_y^2 + E_z^2} = \sqrt{E_x^2 + E_y^2}$ $[\because E_z = 0]$

for uniform plane wave,

$$H = |H| = \sqrt{H_x^2 + H_y^2}$$

$$E = |E| = \sqrt{E_x^2 + E_y^2}$$

Now putting the values of E_x and E_y from eqns (5) & (6)

$$E = \sqrt{(\mu/\epsilon) H_y^2 + \left(\frac{\mu}{\epsilon}\right) H_x^2} = \sqrt{\frac{\mu}{\epsilon}} \sqrt{H_x^2 + H_y^2} = \sqrt{\frac{\mu}{\epsilon}} H$$

$$\boxed{\frac{E}{H} = \sqrt{\frac{\mu}{\epsilon}}}$$

where 'E' and 'H' are the total Electric & H field intensities and this ratio is denoted by η (eta)

$$\boxed{\eta = \frac{E}{H} = \sqrt{\frac{\mu}{\epsilon}}}$$

→ It states that in a plane, Electromagnetic wave has definite ratio between the ~~elect~~ amplitudes of 'E' and 'H', which is equal to the square root of the ratio of perameability to the dielectric constant of the medium,

$$\eta = \frac{E}{H} = \sqrt{\frac{\mu}{\epsilon}} = \frac{\text{volt/mts}}{\text{Amp/mtr}} = \frac{\text{Volt}}{\text{Amp}} = \text{Impedance}.$$

→ It is usually denoted by 'η' or sometimes by 'z' also and is called as 'Intsinsic impedance' or "characteristic impedance" of the ~~ratio~~ non-conducting medium.

→ for free space or vacuume or loss less medium,

$$\eta = \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\,\Omega.$$

→ Now let us focus on relation between E & H. This is ~~de~~ obtain by taking dot product of the two and using the relations we ge

$$E \cdot H = (E_x a_x + E_y a_y + E_z a_z) \cdot (H_x a_x + H_y a_y + H_z a_z)$$

$$= E_x H_x + E_y H_y \qquad [\because E_z = H_z = 0].$$

$$= \eta H_y H_x - \eta H_x H_y = 0 \qquad [\text{from } (5) \,\&\, (6)]$$

$$E \cdot H = 0 \Rightarrow EH\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \boxed{\theta = 90^\circ}$$

$$\because E \neq H \neq 0.$$

proves that in a uniform plane wave E & H are at mutually perpendicular to each other.

further, taking the vector product of $\bar{E}$ & $\bar{H}$ we get

$$E \times H = \begin{vmatrix} a_x & a_y & a_z \\ E_x & E_y & E_z \\ H_x & H_y & H_z \end{vmatrix} = \begin{vmatrix} a_x & a_y & a_z \\ E_x & E_y & 0 \\ H_x & H_y & 0 \end{vmatrix}$$

$$= a_x[0] - a_y[0] + a_z[E_x H_y - E_y H_x] = a_z[E_x H_y - E_y H_x]$$

$$= a_z[\eta H_y^2 - H_x(-\eta H_x)] = a_x \cdot \eta [H_x^2 + H_y^2] = a_x \cdot \eta \cdot H^2$$

Thus cross product of $\bar{E}$ & $\bar{H}$ gives the direction in which the wave progress.

Therefore, we see that the Electric & Magnetic fields of the radiating wave are in time phase in space are proportional to each other are mutually perpendicular and are moving outward along z-direction with a velocity of light.

Sinusoidal Time Variation :

→ Harmonic functions:

Basically in 3 formats.

→ Harmonic function is a 2-Dimensional object entity.

→ $\left.\begin{array}{l} A \sin\theta \\ A \cos\theta \\ A e^{j\theta} \end{array}\right\}$ Domain-2 phase

Domain-1 Amplitude

→ phase is a linear function of variable ($\theta \propto t$).

→ If the phase changes with time linearly called "Time Harmonic".

→ $\theta = \omega t$ ($\theta \propto t$) $\quad \omega \rightarrow$ phase shift constant/time.

$\omega = \theta/t = \frac{2\pi}{T}$.

→ If the phase changes with the space linearly called "space Harmonic".

i.e., $\theta \propto z \Rightarrow \theta = \beta z$ $\quad \beta \rightarrow$ phase shift constant/distance.

$\beta = \theta/z = 2\pi/\lambda$

TL

F(8)

→ $A\sin\omega t \xrightarrow{\text{I-derivative}} A\omega\sin(\omega t+90^\circ) \xrightarrow{\text{II-derivative}} \omega^2 A\sin(\omega t+180^\circ)$

By derivating the sine function also we get the same function but shifted by 90° each time.

→ $A\cos\omega t \xrightarrow{\text{I-derivative}} \omega A\cos(\omega t+90^\circ) \xrightarrow{\text{II-derivative}} \omega^2 A\cos(\omega t+180^\circ)$ ...

→ $Ae^{j\omega t} \xrightarrow{\text{I derivative}} \omega A e^{j(\omega t+90^\circ)} \xrightarrow{\text{II-derivative}} \omega^2 A e^{j(\omega t+180^\circ)} \longrightarrow$ ...

→ In practice, the voltage or current are assumed sinusoidal or co-sinusoidal as the practical generator generates them.

$$E = E_0\cos\omega t \qquad \text{where } \omega = 2\pi f,$$
$$E = E_0\sin\omega t \qquad f \rightarrow \text{the frequency of variation.}$$

It means sinusoidal time factor should be attached to every equatio

→ By using phasor notation, the time factor may be avoided.

→ The electric field intensity vector $E(r,t)$ for time varying may be represent interms of phasor notation as

$$\tilde{E}(r,t) = \text{Real}\{E(r)e^{j\omega t}\}$$

where $\tilde{E}(r,t) \longrightarrow$ Time varying field.

$E(r) \rightarrow$ phasor Quantity

→ The symbol (~) is put over the time varying $E(r,t)$ so that it could be distinguished from phasor Quantity.

propagation constant (γ):—

Consider Maxwells equation derived from faradays Law is

$$\nabla\times E = -\frac{\partial B}{\partial t} = -\mu\frac{\partial H}{\partial t}$$

The general wave equation for any medium is

in E-field: $\nabla^2 E = \mu\sigma\frac{\partial E}{\partial t} + \mu\epsilon\frac{\partial^2 E}{\partial t^2}$ → (1)

in H-field: $\nabla^2 H = \mu\sigma\frac{\partial H}{\partial t} + \mu\epsilon\frac{\partial^2 H}{\partial t^2}$ → (2)

→ Helmholtz Equations

→ In a uniform plane wave, both $\bar{E}$ & $\bar{H}$ fields vary with time, so by the property of phasor when the fields are vary with respect to time.

$$\bar{E} = E_0 e^{j\omega t} \longrightarrow \text{Time Harmonic function.}$$

$$\frac{\partial \bar{E}}{\partial t} = j\omega\left[E_0 e^{j\omega t}\right] = j\omega \bar{E} \longrightarrow ③$$

By taking the second derivative of the time Harmonic function,

$$\frac{\partial^2 \bar{E}}{\partial t^2} = j\omega\left[j\omega E\right] = -\omega^2 \bar{E} \longrightarrow ④$$

substituting eqn's ③ & ④ are in eqn ①, then we get

$$\nabla^2 E = \mu\sigma\left[j\omega \bar{E}\right] + \mu\epsilon\left[-\omega^2 \bar{E}\right]$$

$$= j\mu\sigma\omega \bar{E} - \omega^2 \mu\epsilon \bar{E} = j\omega\mu\sigma \bar{E} + j^2\omega^2\mu\epsilon \bar{E}$$

$$\boxed{\nabla^2 \bar{E} = j\omega\mu\left[\sigma + j\omega\epsilon\right]\bar{E}} \longrightarrow ⑤$$

Similarly for H-field,

By assuming, $H = H_0 e^{j\omega t}$

$$\frac{\partial H}{\partial t} = j\omega H \longrightarrow ⑥$$

$$\frac{\partial^2 H}{\partial t^2} = -\omega^2 H = j^2\omega^2 H \longrightarrow ⑦$$

By substituting eqns ⑥ & ⑦ & eqn ②, we get

$$\nabla^2 H = \mu\sigma\left[j\omega H\right] + \mu\epsilon\left[j^2\omega^2 H\right]$$

$$= j\omega\mu\sigma H + j^2\omega^2\mu\epsilon H$$

$$\boxed{\nabla^2 H = j\omega\mu\left[\sigma + j\omega\epsilon\right]H} \longrightarrow ⑧$$

Eqn's ⑤ & ⑧ are in phasor form. In both these eqns, the term inside the bracket is same. This term represents the properties of the Medium through which the wave is travelling.

These eqns ⑤ & ⑧ can also be written as

$$\nabla^2 E = \gamma^2 E$$
$$\nabla^2 H = \gamma^2 H$$

} These eqns are called 'Helmholtz eqns'.

where $\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$

$$\boxed{\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}} \longrightarrow (9)$$

where $\gamma \rightarrow$ propagation constant.
which decides the coarse of the exponential as the wave advances into the medium.

→ propagation constant composed of a real part 'α' called 'attenuation constant' and in imaginary part 'β' called 'phase constant'.

$$\boxed{\gamma = \alpha + j\beta = \sqrt{j\omega\mu[\sigma + j\omega\varepsilon]}} \longrightarrow (10)$$

→ In general, when the wave travells through medium it gets attenuated that means the amplitude of the medium reduces.

→ Attenuation constant measured in Nepers/mtr. But practically represented in dB.

$$\boxed{\begin{aligned} 1\,Np &= 8.686\,dB \\ 1\,dB &= 0.115\,Np \end{aligned}}$$

→ When the wave travels through the medium, phase change occurs. Such a phase change is expressed by an imaginary part of propagation constant. It is called 'phase shift constant' or 'simply phase constant' β. It is measured in radians/mtr.

→ Given the properties of the medium (σ, μ, ε), we may determine the eqns for the attenuation and phase constants.

$\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon)$

since $\gamma = \alpha + j\beta$

so $(\alpha + j\beta)^2 = j\omega\mu(\sigma + j\omega\epsilon)$

$\alpha^2 - \beta^2 + j2\alpha\beta = j\omega\mu\sigma - \omega^2\mu\epsilon$.

By comparing real & img components on both the sides,

$$\alpha^2 - \beta^2 = -\omega^2\mu\epsilon \longrightarrow (11)$$

$$2\alpha\beta = \omega\mu\sigma \longrightarrow (12)$$

from eqn (12), $\beta = \frac{\omega\mu\sigma}{2\alpha} \longrightarrow (13)$

Substitute eqn(13) in eqn(11), we get

$$\alpha^2 - \frac{\omega^2\mu^2\sigma^2}{4\alpha^2} = -\omega^2\mu\epsilon$$

$$\Rightarrow 4\alpha^4 + 4\omega^2\mu\epsilon\alpha^2 - \omega^2\mu^2\sigma^2 = 0$$

$$\Rightarrow \alpha^4 + \omega^2\mu\epsilon\alpha^2 = \frac{\omega^2\mu^2\sigma^2}{4}$$

$$(\alpha^2)^2 + 2\cdot\alpha^2\cdot\frac{\omega^2\mu\epsilon}{2} + \left(\frac{\omega^2\mu\epsilon}{2}\right)^2 = \left(\frac{\omega^2\mu\epsilon}{2}\right)^2 + \frac{\omega^2\mu^2\sigma^2}{4}.$$

$$\left[\alpha^2 + \frac{\omega^2\mu\epsilon}{2}\right]^2 = \frac{\omega^4\mu^2\epsilon^2}{4}\left[1 + \frac{\sigma^2}{\omega^2\epsilon^2}\right]$$

$$\Rightarrow \alpha^2 + \frac{\omega^2\mu\epsilon}{2} = \frac{\omega^2\mu\epsilon}{2}\sqrt{\left[1 + \frac{\sigma^2}{\omega^2\epsilon^2}\right]}$$

$$\Rightarrow \alpha^2 = \frac{\omega^2\mu\epsilon}{2}\sqrt{1 + \left[\frac{\sigma}{\omega\epsilon}\right]^2} - \frac{\omega^2\mu\epsilon}{2}$$

$$\alpha^2 = \frac{\omega^2\mu\epsilon}{2}\left(\sqrt{1 + \left[\frac{\sigma}{\omega\epsilon}\right]^2} - 1\right)$$

$$\therefore \boxed{\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\epsilon}\right]^2} - 1\right]}} \longrightarrow (14)$$

Nepers/mts

from eqn(12), $\alpha = \frac{\omega\mu\sigma}{2\beta} \longrightarrow (15)$

Substitute eqn(15) in eqn(11)

$$\frac{\omega^2\mu^2\sigma^2}{4\beta^2} - \beta^2 = -\omega^2\mu\varepsilon$$

$$\Rightarrow \omega^2\mu^2\sigma^2 - 4\beta^4 = -4\omega^2\mu\varepsilon\beta^2$$

$$\Rightarrow \beta^4 - \omega^2\mu\varepsilon\beta^2 - \frac{\omega^2\mu^2\sigma^2}{4} = 0.$$

$$\Rightarrow \beta^4 - \omega^2\mu\varepsilon\beta^2 = \frac{\omega^2\mu^2\sigma^2}{4}$$

Add $\left[\frac{\omega^2\mu\varepsilon}{2}\right]^2$ on both the sides

$$\beta^4 - \omega^2\mu\varepsilon\beta^2 + \left(\frac{\omega^2\mu\varepsilon}{2}\right)^2 = \frac{\omega^2\mu^2\sigma^2}{4} + \left(\frac{\omega^2\mu\varepsilon}{2}\right)^2$$

$$\left[\beta^2 - \frac{\omega^2\mu\varepsilon}{2}\right]^2 = \frac{\omega^4\mu^2\varepsilon^2}{4}\left[1 + \frac{\sigma^2}{\omega^2\varepsilon^2}\right]$$

$$\Rightarrow \beta^2 - \frac{\omega^2\mu\varepsilon}{2} = \omega^2\sqrt{\frac{\mu^2\varepsilon^2}{4}\left[1 + \frac{\sigma^2}{\omega^2\varepsilon^2}\right]}$$

$$\Rightarrow \beta^2 - \frac{\omega^2\mu\varepsilon}{2} = \omega^2 \cdot \frac{\mu\varepsilon}{2}\sqrt{\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]}$$

$$\Rightarrow \beta^2 = \frac{\omega^2\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]$$

$$\Rightarrow \boxed{\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]}} \longrightarrow (15)$$

radians/mtr.

Intrinsic Impedance of a Dielectric and Conducting Media

→ The behavior of conducting medium toward plane EM wave is considered from the impedance.

→ Assume that the E & H fields are varying harmonically with time. Hence may be assumed to vary exponentially $e^{j\omega t}$

$$E = E_0 e^{j\omega t}, \quad H = H_0 e^{j\omega t}$$

$$\frac{\partial E}{\partial t} = j\omega E_0 e^{j\omega t} = j\omega E$$

$$\frac{\partial H}{\partial t} = j\omega H_0 e^{j\omega t} = j\omega H$$

Since the wave is propagating in the z-direction with electric & Magnetic field intensities in the 'x' & 'y' directions respectively. The solution of the wave equation in E is written as

$$E = E_0 e^{-\gamma z} a_x .$$

from the Maxwell's eqn, $\nabla \times H = (\sigma + j\omega\epsilon)E$, $\nabla \cdot E = 0$

$\nabla \times E = -j\omega\mu H$ $\nabla \cdot H = 0$.

from $\nabla \times H = (\sigma + j\omega\epsilon)E$

$$\begin{vmatrix} a_x & a_y & a_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ H_x & H_y & H_z \end{vmatrix} = (\sigma + j\omega\epsilon)[E_x a_x + E_y a_y + E_z a_z]$$

Since the wave is propagating along z-direction so E_z & $H_z = 0$, since it is uniform so $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$.

$$\therefore \begin{vmatrix} a_x & a_y & a_z \\ 0 & 0 & \partial/\partial z \\ H_x & H_y & 0 \end{vmatrix} = (\sigma + j\omega\epsilon)[E_x a_x + E_y a_y]$$

$$a_x\left[-\frac{\partial H_y}{\partial z}\right] = (\sigma + j\omega\epsilon)[E_x a_x] \Rightarrow \boxed{\frac{\partial H_y}{\partial z} = -(\sigma + j\omega\epsilon)E_x} \rightarrow (1)$$

$$-a_y\left[-\frac{\partial H_x}{\partial z}\right] = (\sigma + j\omega\epsilon)E_y a_y \Rightarrow \boxed{\frac{\partial H_x}{\partial z} = (\sigma + j\omega\epsilon)E_y} \rightarrow (2)$$

from $\nabla \times E = -j\omega\mu H$.

$$\begin{vmatrix} a_x & a_y & a_z \\ 0 & 0 & \partial/\partial z \\ E_x & E_y & 0 \end{vmatrix} = -j\omega\mu[H_x a_x + H_y a_y]$$

$$\Rightarrow a_x\left[-\frac{\partial E_y}{\partial z}\right] = -j\omega\mu H_x a_x \Rightarrow \boxed{\frac{\partial E_y}{\partial z} = j\omega\mu H_x} \rightarrow (3)$$

$$\Rightarrow -a_y\left[-\frac{\partial E_x}{\partial z}\right] = -j\omega\mu H_y a_y \Rightarrow \boxed{\frac{\partial E_x}{\partial z} = -j\omega\mu H_y} \rightarrow (4)$$

Since $\frac{\partial E_x}{\partial z} = E_0 e^{-\gamma z}[-\gamma] = -\gamma E$

(or) $\frac{\partial E_x}{\partial z} = -\gamma E_x$

$\because \frac{\partial E_x}{\partial z} = -j\omega\mu H_y$

$\therefore \cancel{-}j\omega\mu H_y = \cancel{-}\gamma E_x \Rightarrow \boxed{\frac{E_x}{H_y} = \frac{j\omega\mu}{\gamma}} \longrightarrow (5)$

Similarly $\boxed{\frac{E_y}{H_x} = \frac{j\omega\mu}{\gamma}} \longrightarrow (6)$

for a uniform plane wave, $E_x^2+E_y^2+E_z^2=1$

$H_x^2+H_y^2+H_z^2=1$.

uniform plane

$\therefore$ for a $\wedge$ wave, propagating in $+z$ direction, the impedance η is given by

$$\eta = \frac{|E|}{|H|} = \frac{\sqrt{E_x^2+E_y^2}}{\sqrt{H_x^2+H_y^2}} = \frac{\sqrt{\left(\frac{j\omega\mu}{\gamma}\right)^2 H_y^2 + \left(\frac{j\omega\mu}{\gamma}\right)^2 H_x^2}}{\sqrt{H_x^2+H_y^2}}$$

$$= \left(\frac{j\omega\mu}{\gamma}\right) \times \frac{\sqrt{H_x^2+H_y^2}}{\sqrt{H_x^2+H_y^2}} = \frac{j\omega\mu}{\gamma}.$$

Since $\gamma = j\omega\mu(\sigma+j\omega\varepsilon)$

$$\therefore \boxed{\eta = \frac{j\omega\mu}{j\omega\mu(\sigma+j\omega\varepsilon)} = \sqrt{\frac{j\omega\mu}{\sigma+j\omega\varepsilon}}.}$$

This is the intrinsic impedance for any Medium.

Wave propagation in Conducting Medium :—

The solution for a uniform plane wave propagating in $+z$-direction, the wave eqn becomes

$$\frac{\partial^2 E_x}{\partial z^2} = \gamma^2 E_x \quad \& \quad \frac{\partial^2 E_y}{\partial z^2} = \gamma^2 E_y$$

The solution for the above wave equation is

$$E_x(z) = A e^{-\gamma z} + B e^{+\gamma z}.$$

Let us assume it is for time varying fields,

$$E(z,t) = \text{Real}\left\{ A e^{-\gamma z} \cdot e^{j\omega t} + B e^{+\gamma z} \cdot e^{j\omega t} \right\} \qquad \text{since } \gamma = \alpha + j\beta.$$

$$= \text{Real}\left\{ A e^{-(\alpha+j\beta)z} e^{j\omega t} + B e^{(\alpha+j\beta)z} e^{j\omega t} \right\}$$

$$= \text{Real}\left\{ \underbrace{A e^{-\alpha z} \cdot e^{+j[\omega t - \beta z]}}_{\text{forward wave}} + \underbrace{B e^{\alpha z} e^{j(\omega t + \beta z)}}_{\text{Reflected wave}} \right\}$$

Assume,

If there is no reflecting surface present. so

$$E(z,t) = A e^{-\alpha z} \cos(\omega t - \beta z)$$

$$= e^{-\alpha z} A \cos(\omega t - \beta z).$$

This shows that in conducting medium, A progressing wave in $+z$-direction is attenuated by a factor $e^{-\alpha z}$ and it is also retarded linearly in phase.

Wave propagation in Loss less [(or) perfect Dielectric] Medium:—

The wave equation in general is given by

$$\nabla^2 E = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

for time Harmonic function, $E = E_0 e^{j\omega t}$

$$\frac{\partial E}{\partial t} = j\omega E, \quad \frac{\partial^2 E}{\partial t^2} = -\omega^2 E.$$

$$\therefore \nabla^2 E = -\mu_0 \varepsilon_0 \omega^2 E \Rightarrow \boxed{\nabla^2 E + \omega^2 \mu_0 \varepsilon_0 E = 0}.$$

This is the vector Helmholtz equation.

$$\nabla^2 E + \beta^2 E = 0.$$

Now for a uniform plane wave propagating in $+z$-direction, there are no x & y-components and so E_z & H_z are zero. Thus the wave eqn reduces to

$$\frac{\partial^2 E_x}{\partial z^2} = -\beta^2 E_x, \quad \frac{\partial^2 E_y}{\partial z^2} = -\beta^2 E_y$$

Similar equations can be written for H also.

Now by taking the y-component of $\bar{E}$ only, the solution of the wave equation can be written as

$$(D^2+\beta^2)\bar{E}_y = 0 \Rightarrow \bar{E}_y = A_1 e^{-j\beta z} + A_2 e^{+j\beta z}.$$

$\because D = \pm j\beta$ where A_1 & A_2 are arbitrary constants.

This is the solution in phasor Quantity.

→ The solution interms of time varying may be written as

$$\bar{E}_y(z,t) = \text{Real}\left\{ \bar{E}_y(z)\, e^{j\omega t}\right\} = \text{Real}\left\{ (A_1 e^{-j\beta z} + A_2 e^{j\beta z}).e^{j\omega t}\right\}$$

$$= \text{Real}\left\{ A_1 e^{j(\omega t-\beta z)} + A_2 e^{j(\omega t+\beta z)}\right\}$$

$$\boxed{\bar{E}_y(z,t) = A_1 \cos(\omega t-\beta z) + A_2 \cos(\omega t+\beta z).} \longrightarrow (A)$$

This is obtained by taking the real part of the complex equation. It is the solution in time-varying field,

→ Wave velocity is given by

$$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = C.$$

Since the medium is loss less i.e., perfect, so attenuation constant $\alpha = 0$.

→ From the above eqn(A), it represents two waves, of which the first term is travelling in +z-direction and the second term in −z-direction. If, however $A_1 = A_2$, the two waves combined and form a simply standing wave which stands and doesn't progress.

Conductors & Dielectrics :—

→ In Electromagnetics, materials are generally classified into two categories, the conductors and Dielectrics or Insulators.

→ from Maxwell's eqn, $\nabla \times H = J + \frac{\partial D}{\partial t} = J_c + J_D = \sigma E + j\omega\epsilon E$

$J_c = \sigma E$ = Conduction Current density

$J_D = j\omega\epsilon E$ = Displacement Current density

→ The ratio of Conduction Current density to Displacement Current density is given by

$$\left|\frac{J_c}{J_D}\right| = \frac{|\sigma E|}{|j\omega \epsilon E|} = \frac{\sigma}{\omega \epsilon}$$

Therefore $\left|\frac{J_c}{J_D}\right| = \frac{\sigma}{\omega\epsilon} = 1$, considered to mark the dividing line between Conductors & Dielectrics.

→ If $\sigma \neq 0$, we may arbitrarily define three conditions as follows

(1) $\frac{\sigma}{\omega\epsilon} << 1$ leads to $\sigma << \omega\epsilon$ i.e., good dielectrics.

(2) $\frac{\sigma}{\omega\epsilon} = 1$ leads to $\sigma \cong \omega\epsilon$ i.e. Quasi Conductors.

(3) $\frac{\sigma}{\omega\epsilon} >> 1$ leads to $\sigma >> \omega\epsilon$ i.e., good Conductors.

→ The Medium for which displacement Current is much greater than conduction Current, is acts as "Dielectrics".

→ The Medium for which conduction Current is much greater than displacement Current, is acts as "Conductor".

→ If $\sigma = 0$, the medium is perfect or Lossless Dielectric Medium.

→ The properties of the Material is normally given interms of the dielectric constant ϵ' and $\frac{\sigma}{\omega\epsilon}$.

→ Ex:- for Good Conductor is copper.

Even at higher frequencies up to 30 GHz, $\frac{\sigma}{\omega\epsilon} = 3.5\times10^{8}$.

Example for good dielectrics or insulator is Mica.

At Audio or radio freq's, $\frac{\sigma}{\omega\epsilon} = 0.0002$.

→ for Good conductors, σ & ϵ are independent of frequency but for dielectrics i.e., σ & ϵ are functions of frequency.

→ Under these Circumstances the ratio $\frac{\sigma}{\omega\epsilon}$ is called 'Dissipation factor (D) of the dielectrics.

→ Most Materials are either good conductor or good insulator.

*) Only the earth occupies an intermediate position throughout most of the

More specifically,

for Dielectrics, $\frac{\sigma}{\omega\varepsilon} < 0.01$ (or) $\frac{1}{100}$

for Quasi-conductors, $\frac{1}{100} < \frac{\sigma}{\omega\varepsilon} < 100$

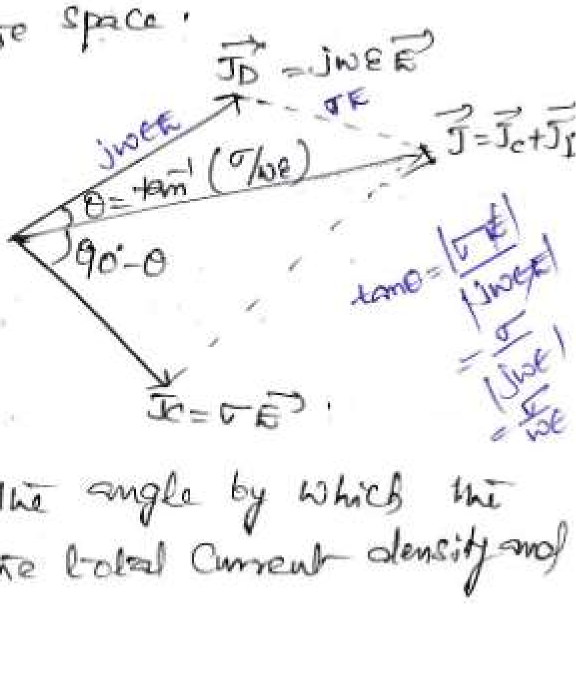
for Conductors, $\frac{\sigma}{\omega\varepsilon} > 100$.

→ The term $\frac{\sigma}{\omega\varepsilon}$ is some times referred to as the "Loss Tangent" since the ratio of conduction Current density to displacement Current density is given by

$$\frac{J_c}{J_D} = \frac{\sigma}{j\omega\varepsilon}.$$

This indicates that two vectors are 90° out of phase in time and are in the same direction in the space.

→ The displacement Current density leads conduction Current density by $\pi/2$ or 90°.



→ The angle 'θ' may be identified as the angle by which the displacement Current density leads the total Current density and

$$\tan\theta = \frac{\sigma}{\omega\varepsilon}$$

This relationship has lead to the name of "Loss Tangent" for $\frac{\sigma}{\omega\varepsilon}$

Wave propagation in perfect (or Loss less) Dielectrics:—

→ for a perfect Dielectric, $\sigma = 0$, $\mu = \mu_r\mu_0$ & $\varepsilon = \varepsilon_0\varepsilon_r$.

→ for the perfect dielectric, if conduction is zero ($\sigma = 0$) i.e., the medium is called "Loss less Medium".

→ The analysis for a wave prop through perfect Dielectrics are similar to the wave propagation through free space.

→ The velocity of wave propagation is

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{(\mu_0\mu_r)(\varepsilon_0\varepsilon_r)}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}} \text{ m/sec}$$

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{\omega}{\beta} \text{ m/sec}$$

→ propagation constant is given by $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$ m^{-1}

→ for a perfect Dielectric, $\sigma = 0$, $\varepsilon = \varepsilon_0\varepsilon_r$, $\mu = \mu_0\mu_r$

$$\therefore \gamma = \pm j\omega\sqrt{\mu\varepsilon} \; m^{-1}$$

Since $\gamma = \alpha \pm j\beta$,

so $\alpha = 0$, $\beta = \omega\sqrt{\mu\varepsilon}$ rad/m

→ The phase constant of the perfect Dielectric is given by

$$\beta = \omega\sqrt{\mu\varepsilon} \text{ rad/m}$$

→ No attenuation, only phase shift in nature.

→ Intrinsic impedance is given by,

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}} = \eta_0\sqrt{\frac{\mu_r}{\varepsilon_r}} = 377\sqrt{\frac{\mu_r}{\varepsilon_r}}\,\Omega.$$

$$= \frac{377}{\sqrt{\varepsilon_r}}\,\Omega.$$

Since $\sigma = 0$ & $\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$.

→ Phase velocity $v_p = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu_0\varepsilon_0\varepsilon_r\mu_r}} = \frac{3\times10^8}{\sqrt{\varepsilon_r}}$ m/sec $[\because \mu_r = 1]$.

Wave propagation in Good Dielectrics :—

[(or) Lossy Dielectrics]

→ conductivity is not zero for lossy Dielectrics, but it is very small.

→ The condition for perfect practical or lossy Dielectric is $\frac{\sigma}{\omega\varepsilon} << 1$.

Since $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$

$$= \sqrt{j\omega\mu \cdot j\omega\varepsilon\left[1+\frac{\sigma}{j\omega\varepsilon}\right]} = j\omega\sqrt{\mu\varepsilon}\left[1+\frac{\sigma}{j\omega\varepsilon}\right]^{1/2}$$

By Binomial series Expansion,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \quad \text{where } |x|<1.$$

So let $x = \frac{\sigma}{j\omega\varepsilon}$ & $n = 1/2$, then neglecting higher order terms

$$\left[1+\frac{\sigma}{j\omega\varepsilon}\right]^{1/2} = 1+\frac{1}{2}\left[\frac{\sigma}{j\omega\varepsilon}\right].$$

$$\therefore \boxed{\gamma = j\omega\sqrt{\mu\varepsilon}\left[1+\frac{1}{2}\frac{\sigma}{j\omega\varepsilon}\right] = j\omega\sqrt{\mu\varepsilon} + \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}}.$$

Since $\gamma = \alpha + j\beta$

By comparing real & imaginary components, we get—

$$\boxed{\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}} \text{ NP/m}, \quad \boxed{\beta = \omega\sqrt{\mu\varepsilon}} \text{ rad/m}.$$

→ The intrinsic impedance 'η' is given by

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon\left[1+\frac{\sigma}{j\omega\varepsilon}\right]}} = \sqrt{\frac{\mu}{\varepsilon}}\left[1+\frac{\sigma}{j\omega\varepsilon}\right]^{-1/2}$$

from Binomial Expansion

$$\boxed{\eta = \sqrt{\frac{\mu}{\varepsilon}}\left[1-\frac{1}{2}\left(\frac{\sigma}{j\omega\varepsilon}\right)\right]}$$

→ The velocity of the propn of wave in the dielectric is then

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}\left[1+\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2+\cdots\right]} = v_0\left[1+\frac{\sigma^2}{8\omega^2\varepsilon^2}\right]^{-1}$$

$$\boxed{v_p = v_0\left[1-\frac{\sigma^2}{8\omega^2\varepsilon^2}\right.} + \cdots\Big]$$

→ The effect of a small amount of Loss is to reduce slightly the velocity of propn of the wave.

$\rightarrow \frac{\sigma}{\omega\varepsilon} = \frac{mho/mtr}{1/sec * farad/mtr} = \frac{sec}{ohm * farad} = \frac{sec}{sec} = $ unitless.

$[\because \tau = RC]$.

Wave propagation in Good Conductors :-

$\rightarrow$ The conductivity is of the order of 10^7 ℧/m in the good conductor like in Cu etc.

$\rightarrow$ for Good conductor, $\frac{\sigma}{\omega\varepsilon} >> 1$ (or) $\sigma >> \omega\varepsilon$.

Since $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$

Since $\sigma >> \omega\varepsilon$, so we can neglect imaginary part

$\therefore \gamma = \sqrt{j\omega\mu\sigma} = \sqrt{\omega\mu\sigma}\,\angle 45^\circ = \sqrt{\omega\mu\sigma}\, e^{j45^\circ}$

$= \sqrt{\omega\mu\sigma}\,[\cos 45^\circ + j\sin 45^\circ] = \sqrt{\omega\mu\sigma}\,\left[\frac{1}{\sqrt{2}} + j\cdot\frac{1}{\sqrt{2}}\right]$

Since $\gamma = \alpha + j\beta$.

So $\boxed{\alpha = \sqrt{\frac{\omega\mu\sigma}{2}}}$ Np/m, $\boxed{\beta = \sqrt{\frac{\omega\mu\sigma}{2}}}$ rad/mtr.

$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$ (or) $\boxed{\alpha = \beta = \sqrt{\pi f \mu\sigma}}$.

$\rightarrow$ Intrinsic impedance (η) is given by

$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$ $\quad \because \sigma >> \omega\varepsilon$

So $\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}}\,\angle 45^\circ = \sqrt{\frac{\omega\mu}{\sigma}}\, e^{j45^\circ}$

$= \sqrt{\frac{\omega\mu}{\sigma}}\,[\cos 45^\circ + j\sin 45^\circ]$

$= \sqrt{\frac{\omega\mu}{\sigma}}\,[1/\sqrt{2} + j\,1/\sqrt{2}]$

Since $\eta = R + jX$.

So $\boxed{R = \sqrt{\frac{\omega\mu}{2\sigma}}}\ \Omega$; $\boxed{X = \sqrt{\frac{\omega\mu}{2\sigma}}}\ \Omega$.

$$\boxed{\eta = \sqrt{\frac{\pi f \mu}{\sigma}}\left[1 + j\cdot 1\right]}$$

$$\boxed{R = X = \sqrt{\frac{\omega\mu}{2\sigma}}}\ \Omega$$

** Since α & β are very large hence there is heavy attenuation therefore EM waves cannot propagate through the good conductors.

As η's real & imaginary parts are equal so that the

As η's real & imaginary parts are equal.

γ's real and imaginary parts are equal.

As η's phase is 45°, E- & H- fields are out of phase by 45° in conductors. which is the maximum possible for any EM wave in any medium.

Skin Depth (or) Depth of penetration :—

→ It is the distance travelled by a wave where the amplitude decays to $1/e$ times of its initial value.

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}} = \frac{1}{\sqrt{\pi f \mu\sigma}}$$

→ It may be defined as that depth in which the wave has been attenuated by an amount $(1/e)$ or 37% approximately of its initial value.

→ Intrinsic impedance of a good conductor interms of skin depth 'δ' is $\eta = \frac{\sqrt{2}}{\sigma\delta} \angle 45^\circ\ \Omega$

→ Velocity of propn, $v = \sqrt{\frac{2\omega}{\mu\sigma}} = \omega\delta$ m/sec

→ Wavelength $\lambda = \frac{2\pi}{\beta} = 2\pi\delta$ mts.

Skin Resistance: —

$$R_S = \text{Real}(\eta) \Rightarrow \boxed{R_S = \sqrt{\frac{\omega\mu}{2\sigma}} = \sqrt{\frac{\pi f\mu}{\sigma}}}$$

→ Relation between skin Depth & skin Resistance is

$$\boxed{R_S = \frac{\alpha}{\sigma}}$$

Polarization of a wave:

→ Polarization always refers to the orientation of E-field with respect to Earth Surface.

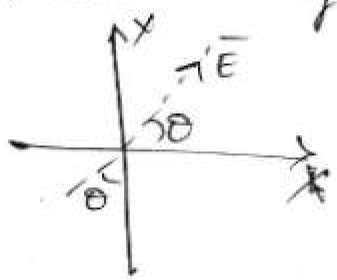
→ polarization of wave is defined as the electric field at a given point as a function of time.

→ Depending on the orientation of Electric field vector, there are three types of polarization techniques. they are Linear, Circular and Elliptical polarizations.

→ Linear polarization: —

→ If E-field remains along a straight line as a function of time at some point in the Medium.

→ If the wave has only one E-field planar component (or) two planar components both are in inphase, the wave is said to be "Linearly polarized wave".



$y = \left(\frac{E_{y0}}{E_{x0}}\right)x$ → Linearly

$y = mx + c$. ↑ Locus of points

UTL-V (16). → Linear polarization of a wave is again of three types: 45

*» Horizontal polarization

*» Vertical polarization

*» Theta "

→ When a wave travels in z-direction with E&H fields lieing in xy-plane.

*» If $E_y = 0$ & E_x is present, it is said to be x-polarized or horizontally polarized wave.

*» If E_y is present & $E_x = 0$, the wave is said to be y-polarized or vertically polarized wave.

*» If E_x & E_y are present and if they are in inphase then the wave is said to be Theta (θ) polarized wave.

Circular polarization:—

→ A wave is said to be circularly polarized when the electric field traces a circle.

→ If the Electric field planar components are out-of phase by 90° & have equal amplitude, then the wave is said to be "circularly polarized wave".

→ Circulary polarized wave has two types:

⇒ Right circularly polarized wave

⇒ Left circularly polarized wave.

→ If the Left-hand thumb points towards the propagation & the closed fingers along the advancing times, the wave is said to be "left circularly polarized wave", otherwise it is a "Right circularly polarized wave".

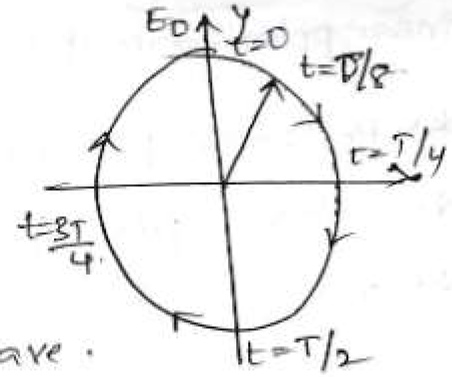
Ex:— $E(z,t) = E_0 \sin(\omega t - \beta z) a_x + E_0 \cos(\omega t - \beta z) a_y$

Sol: Study the E-field trace on the propn constant plane for various advancing times.

At $t=0$, $E_x=0$, $E_y=E_0$.

$t=T/8$, $E_x=E_0/\sqrt{2}$, $E_y=\frac{E_0}{\sqrt{2}}$

$t=T/4$, $E_x=0$, $E_y=0$.



~~Right~~ Left circularly polarized wave.

→ if

$$\left.\begin{array}{l} E_y \to \sin \\ E_z \to \cos \\ \text{prop} \to -x \end{array}\right\} RCP \qquad \left.\begin{array}{l} E_y \to \sin \\ E_z \to \cos \\ \text{prop}^n \to +x \end{array}\right\} LCP$$

$$\left.\begin{array}{l} E_y \to \cos \\ E_z \to \sin \\ \text{prop}. \to +x \end{array}\right\} RCP \qquad \left.\begin{array}{l} E_y \to -\sin \\ E_z \to \cos \\ \text{prop}. \to +x \end{array}\right\} RCP$$

Elliptical polarization:

→ If the planar components are out-of phase by any amount not other than $90°$ and $0°$, and if the amplitudes or Magnitudes are of any (Magnitude) value, then this technique called "Elliptical polarization".

→ Axial Ratio = $\frac{\text{Major Axis}}{\text{Minor Axis}}$ AR Range = $[1, \infty]$.

AR = 1 → Circular

AR = ∞ → Linear

1 < AR < ∞ → Elliptical.

Examples

→ $E(x,t) = 25\sin(\omega t+4x)(a_y+6a_z)$ → Linearly polarised wave

→ $E(x,t) = 25\sin(\omega t+4x+60°)(a_y+6a_z)$ → Linear "

→ $E(x,t) = 25\sin(\omega t+4x)(a_y+6ja_z)$ → Elliptically "

→ $E(x,t) = 25\sin(\omega t+4x)(a_y+ja_z)$ → Circularly "

→ $E(x,t) = 25\sin(\omega t+4x)(a_y+6e^{j60}a_z)$ → Elliptically "

→ $E(x,t) = 25\sin(\omega t+4x)(a_y+e^{j60°}a_z)$ → Elliptically "

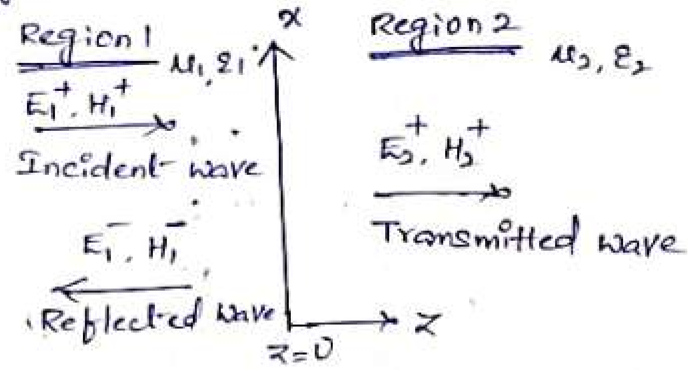
→ $E(x,t) = 25\sin(\omega t+4x)[a_x+(1+j)a_z]$ → Elliptically "

→ $E(x,t) = 25\sin(\omega t+4x)[(1+j)a_y+(1-j)a_z]$ → Circularly "

Reflection and Refraction

→ The process of reflection/refraction occurs due to the discontinuity encountered at the interface of two media.

→ When the Electromagnetic waves travels in the dispersive media, some parameters of the media such as σ, μ, ϵ that affects the propagation of the wave. For example, permittivity varies frequency that affects the propagation of the wave.



→ When a plane wave travelling from one medium to another medium, it is partly reflected and partly transmitted.

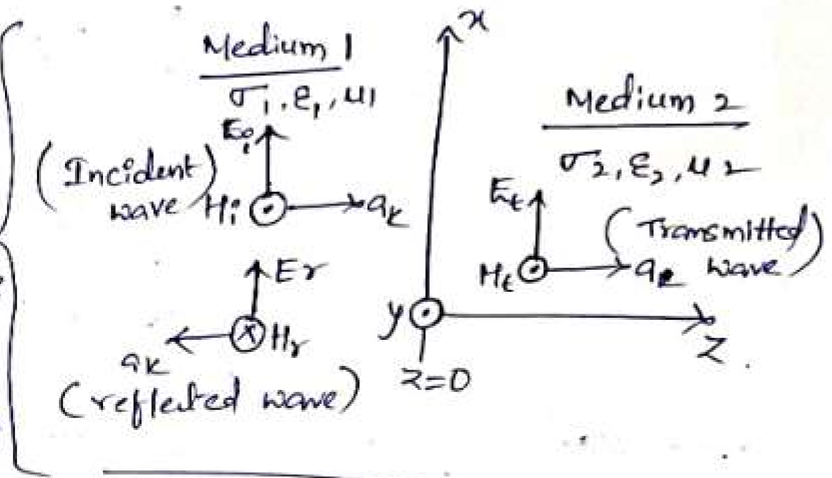
→ The proportion of the incident wave that is reflected (or) transmitted depends on the parameters such as σ, μ & ϵ of the two media ~~total~~ involved.

→ When a plane EM wave is incident on a planar boundary between two homogenous media, the scattered waves are also plane waves. one of these waves is radiated back into the half-space of the incident wave; this wave is known as the "reflected wave". There is also a wave in the other half-space (except in the case of a perfect conductor), propagating generally in a different direction from the incident wave; it is therefore called the 'refracted or transmitted wave'.

Suppose that a plane wave propagating along the +z-direction is entering on to the boundary separating two medias via either Normal incidence or oblique incidence.

Normal Incidence :—

Assume that a plane wave propagating along +z-direction is incident normally on the boundary $z=0$ between medium 1 ($z<0$) characterized by $\sigma_1, \epsilon_1, \mu_1$ and medium 2 ($z>0$) characterized by $\sigma_2, \epsilon_2, \mu_2$, as shown in figure.



Incident Wave:—

(E_i, H_i) is traveling along $+a_z$ in medium 1. If we suppress the time factor $e^{j\omega t}$ and assume that

$$E_i(z) = E_{i0}\, e^{-\gamma_1 z} a_x \quad \text{then} \quad H_i(z) = H_{i0}\, e^{-\gamma_1 z} a_y = \frac{E_{i0}}{\eta_1} e^{-\gamma_1 z} a_y$$

Reflected Wave:—

(E_r, H_r) is traveling along $-a_z$ in medium 1. If

$$E_r(z) = E_{r0}\, e^{\gamma_1 z} a_x, \text{ then } H_r(z) = H_{r0}\, e^{\gamma_1 z}(-a_y) = -\frac{E_{r0}}{\eta_1} e^{\gamma_1 z} a_y$$

Transmitted Wave:—

(E_t, H_t) is travelling along $+a_z$ in medium 2. If

$$E_t(z) = E_{t0}\, e^{-\gamma_2 z} a_x \quad \text{then} \quad H_t(z) = H_{t0}\, e^{-\gamma_2 z} a_y = \frac{E_{t0}}{\eta_2} e^{-\gamma_2 z} a_y$$

Where E_{i0}, E_{r0} & E_{t0} are the magnitudes of the incident, reflected and transmitted electric fields at $z=0$.

It is noticed that the total field in medium 1 comprises both the incident and reflected fields, whereas medium 2 has only the transmitted field, that is

$$E_1 = E_i + E_r \quad , \quad E_2 = E_t$$

$$H_1 = H_i + H_r \quad , \quad H_2 = H_t$$

At the interface $z=0$, the boundary conditions require that the tangential components of $\vec{E}$ and $\vec{H}$ fields must be continuous. Since the waves are transverse, E and H fields are entirely tangential to the interface. Hence at $z=0$, $E_{1tan} = E_{2tan}$ and $H_{1tan} = H_{2tan}$ imply that

$$E_i(0) + E_r(0) = E_t(0) \implies E_{i0} + E_{r0} = E_{t0}$$

$$H_i(0) + H_r(0) = H_t(0) \implies \frac{1}{\eta_1}(E_{i0} - E_{r0}) = \frac{E_{t0}}{\eta_2}$$

$$\therefore \frac{1}{\eta_1}(E_{i0} - E_{r0}) = \frac{E_{i0} + E_{r0}}{\eta_2} \implies \frac{E_{i0}}{\eta_1} - \frac{E_{r0}}{\eta_1} = \frac{E_{i0}}{\eta_2} + \frac{E_{r0}}{\eta_2}$$

$$\implies E_{i0}\left[\frac{1}{\eta_1} - \frac{1}{\eta_2}\right] = E_{r0}\left[\frac{1}{\eta_1} + \frac{1}{\eta_2}\right] \implies E_{i0}\left[\frac{\eta_2 - \eta_1}{\eta_1\eta_2}\right] = E_{r0}\left[\frac{\eta_2 + \eta_1}{\eta_1\eta_2}\right]$$

$$\implies \boxed{E_{r0} = \left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right) E_{i0}}$$

and Γ = reflection Coefficient $= \dfrac{E_{r0}}{E_{i0}} = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_2}$

or $\boxed{E_{r0} = \Gamma E_{i0}}$

and $T = 1 + \Gamma = 1 + \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \dfrac{2\eta_2}{\eta_2 + \eta_1}$

or $\boxed{E_{t0} = T E_{i0}}$

*) $1 + \Gamma = T$

*) Both Γ and T are dimensionless and may be complex.

*) $0 \leq |\Gamma| \leq 1$

(i) perfect Dielectric to perfect Conductor interface

consider a special case when medium 1 is perfect dielectric (lossless, $\sigma_1=0$) and medium 2 is a perfect conductor ($\sigma_2 \simeq \infty$). for this case, $\eta_2=0$; hence $\Gamma=-1$ and $T=0$, showing that the wave is totally reflected. This should be expected because fields in a perfect conductor must vanish, so there can be no transmitted wave ($E_2=0$). The totally reflected wave combines with the incident wave to form a "standing wave". The standing wave consists of two travelling waves (E_i and E_r) of equal amplitudes but in opposite directions.

$$\therefore\ E_1 = E_i + E_r = \left(E_{i0}\, e^{-\gamma_1 z} + E_{r0}\, e^{\gamma_1 z}\right) a_x$$

But $\Gamma = \dfrac{E_{r0}}{E_{i0}}$, hence

$$E_1 = -E_{i0}\left(e^{j\beta_1 z} - e^{-j\beta_1 z}\right) a_x \Rightarrow E_1 = -2j E_{i0} \sin\beta_1 z\, a_x$$

thus;

$$E_1 = Re\left(E_1 e^{j\omega t}\right)$$

or

$$\boxed{E_1 = 2E_{i0} \sin\beta_1 z \sin\omega t\, a_x}$$

By taking similar steps, it can be shown that the magnetic field component of the wave is

$$\boxed{H_1 = \frac{2E_{i0}}{\eta_1} \cos\beta_1 z \cos\omega t\, a_y}$$

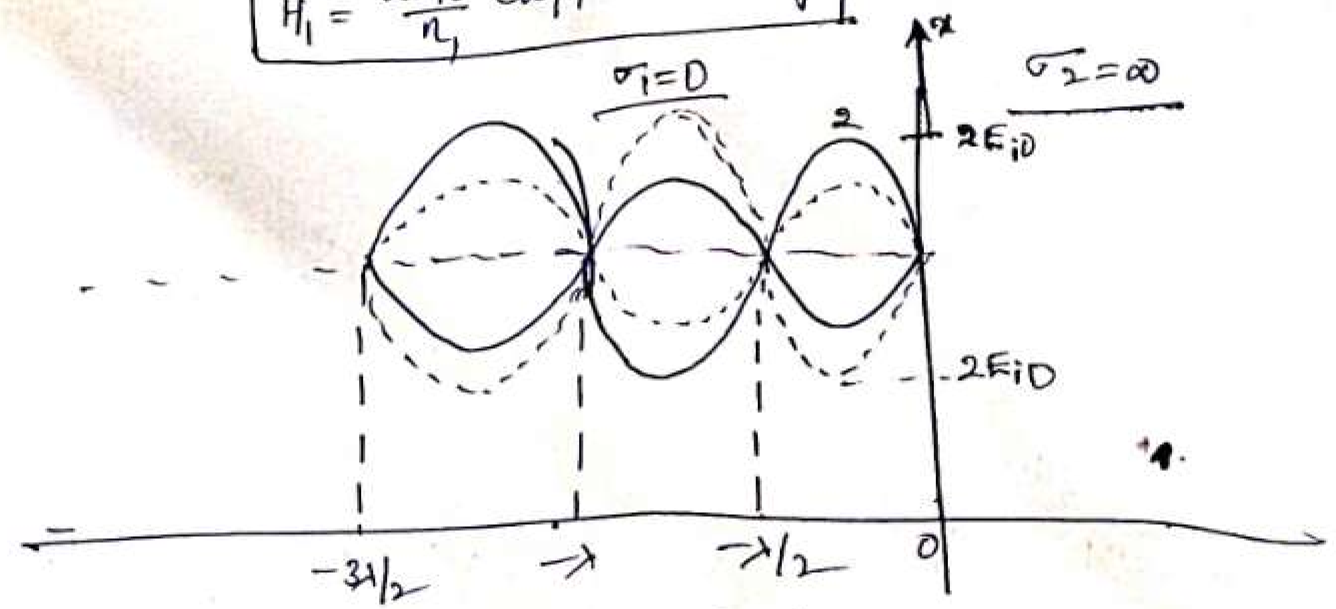


Fig: Standing waves $E = 2E_{i0} \sin\beta_1 z \sin\omega t\, a_x$; Curves 0, 1, 2, 3, 4 . . . are, respectively, at times $t = 0, T/8, T/4, 3T/8, T/2 \ldots$; $\lambda = 2\pi/\beta_1$.

(ii) when $\eta_2 > \eta_1$:-

If $\eta_2 > \eta_1$, $\Gamma > 0$.

A standing wave is formed in medium1 but there is also a transmitted wave in medium2. However, the incident and reflected waves have amplitudes that are not equal in magnitude. It can be shown that a relative ~~minima~~ maxima of $|E_1|$ occurs at

$$-\beta_1 z_{max} = n\pi$$

or $$z_{max} = \frac{-n\pi}{\beta_1} = \frac{-n\lambda_1}{2} \quad \text{where } n = 0, 1, 2 \ldots$$

and the ~~maxima~~ minima of $|E_1|$ occurs at

$$-\beta_1 z_{min} = (2n+1)\pi/2$$

or $$z_{min} = -\frac{(2n+1)\pi}{2\beta_1} = -\frac{(2n+1)}{4}\lambda_1, \quad n = 0, 1, 2, \ldots$$

(iii) If when $\eta_2 < \eta_1$:-

If $\eta_2 < \eta_1$, $\Gamma < 0$.

In this case, the locations of $|E_1|$ maximum are occured at

$$-\beta_1 z_{max} = (2n+1)\pi/2$$

or $$z_{max} = \frac{-(2n+1)}{4}\lambda_1, \quad n = 0, 1, 2, \ldots$$

The locations of $|E_1|$ Minimas are occured at

$$-\beta_1 z_{min} = n\pi$$

or $$z_{min} = -\frac{n\pi}{\beta} = -\frac{n\lambda_1}{2}, \quad n = 0, 1, 2, \ldots$$

*> $|H_1|$ minimas occurs whenever there is $|E_1|$ maximum and vice versa.

*> The transmitted wave in medium 2 is a purely travelling wave and consequently there are no maxima and minima in this region

Therefore, the standing wave ratio 'S' is

$$\boxed{S = \frac{|E_1|_{max}}{|E_1|_{min}} = \frac{|H_1|_{max}}{|H_1|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}}$$

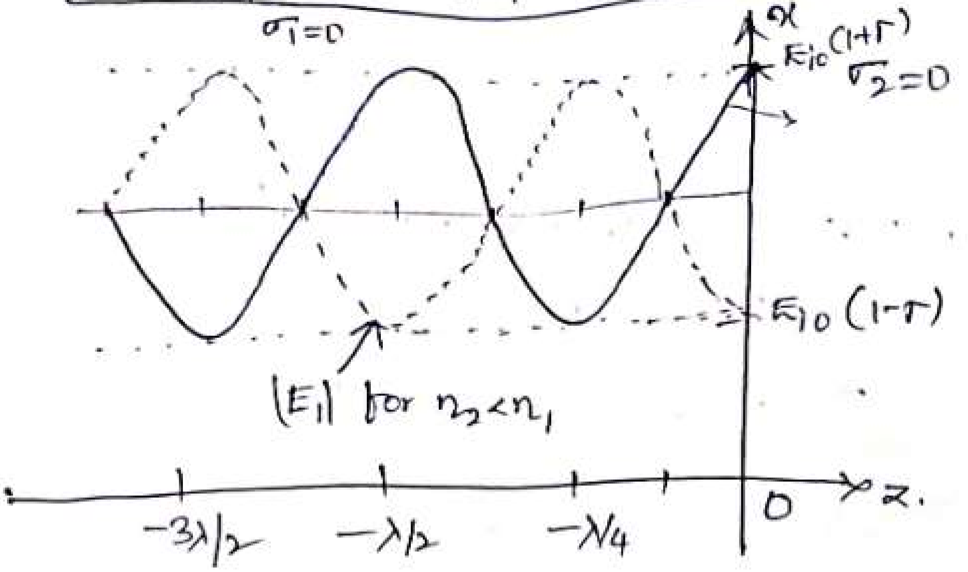


fig: standing waves due to reflection at an interface between two lossless media; $\lambda = 2\pi/\beta_1$.

$$|\Gamma| = \frac{S-1}{S+1}$$

Since $|\Gamma| \leq 1$, and hence $1 \leq S \leq \infty$.

$$\boxed{S(dB) = 20\log_{10} S}$$

Reflection of a plane wave at oblique incidence:

Case (iv): — When $\eta_2 = 0$ in a $\eta_1 > \eta_2$: —

The medium 1 is perfect Dielectric ($\sigma_1 = 0$) and medium 2 is perfectly conducting ($\sigma_2 = \infty$).

$$\therefore \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}, \quad \eta_2 = 0, \quad \gamma_1 = \sqrt{j\omega\mu_1(j\omega\varepsilon_1)} = j\omega\sqrt{\mu_1\varepsilon_1} = j\beta_1$$

Since $T = \frac{2\eta_2}{\eta_1+\eta_2}$, $T = 1+\Gamma$

$$\boxed{T = 0 \text{ and } \Gamma = -1}$$

Hence, the wave is not transmitted into the medium 2, it gets reflected entirely from the interface to the medium 1.

$$\therefore E_1 = E_{i0} e^{-j\beta_1 z} a_x - E_{i0} e^{j\beta_1 z} a_x = -2j E_{i0} \sin\beta_1 z \, a_x$$

$$E_1(t) = \mathrm{Real}\left[-2j E_{i0} \sin\beta_1 z \, e^{j\omega t}\right] a_x = 2E_{i0} \sin\beta_1 z \sin\omega t \, a_x$$

Similarly for the magnetic field in region 1, we can show that

$$\boxed{H_1(t) = \frac{2E_{i0}}{\eta_1} \cos\beta_1 z \cos\omega t \, a_y}$$

The wave in medium 1 thus becomes a standing wave due to the superposition of a forward travelling wave and a backward travelling wave.

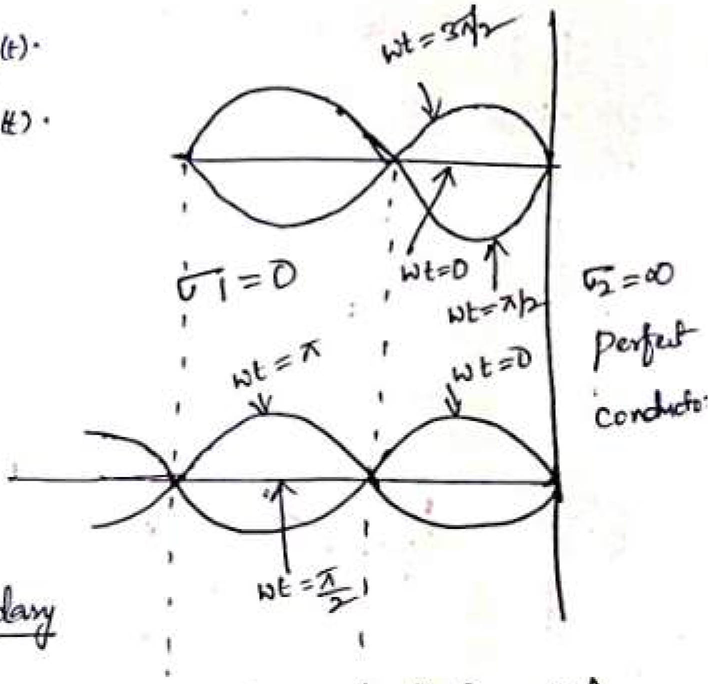
→ Minimas of $E_1(t)$ are the maximas of $H_1(t)$.

→ Maximas of $E_1(t)$ are the minimas of $H_1(t)$.

occur at $\beta_1 z = -n\pi \Rightarrow z = -n\lambda/2$

occur at $\beta_1 z = -(2n+1)\pi/2$ or

$z = -(2n+1)\lambda/4$, $n = 0, 1, 2 \ldots$



(V) Normal Incidence on a plane dielectric Boundary

If the medium 2 is not a perfect conductor (i.e. $\sigma_2 \neq \infty$) partial reflection will result. There will be reflected wave in the medium 1 and a transmitted wave in the medium 2. Because of the reflected wave, standing wave is formed in medium 1. Hence

$$E_1 = E_{i0}\left(e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z}\right) a_x.$$

Reflection of a plane wave at oblique incidence:—

Plane of Incidence: the plane containing the propagation of incident wave and unit normal vector to the boundary.

→ The oblique incidence case is usually apply to a lossless medium

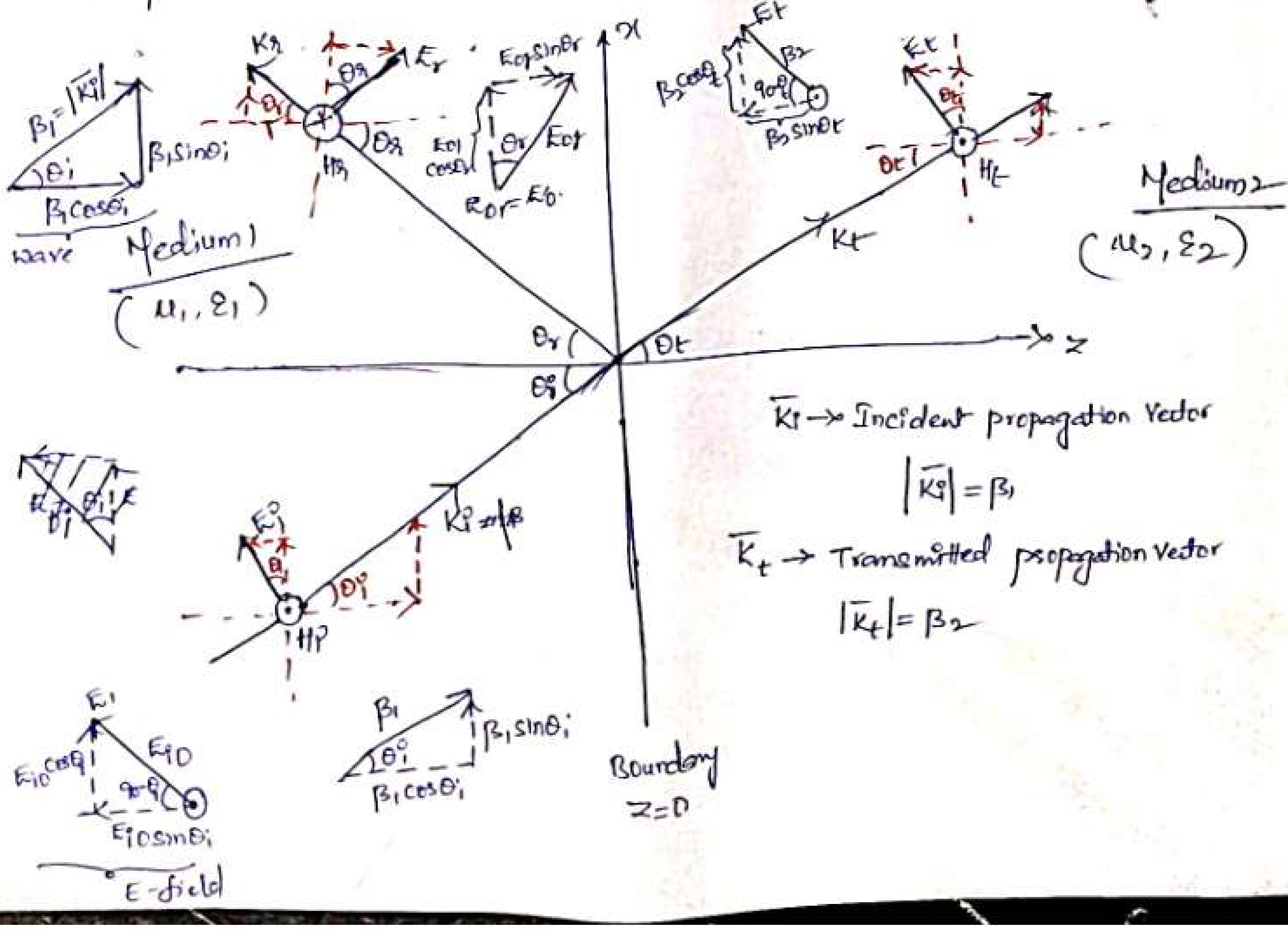
→ These are two types of oblique incidence.

(i) parallel polarization: — If E-field of uniform plane wave is parallel to the plane of incident. It is called 'parallel polarization'.

(ii) perpendicular polarization:— If the E-field of uniform plane wave perpendicular to the plane of incidence. It is called 'perpendicular polarization'.

(i) parallel polarization:

The E-field is now lies in the incidence plane and will have x and z components, whereas the H-field is perpendicular to the plane of incidence.



$\overline{K_i} \rightarrow$ Incident propagation vector

$|\overline{K_i}| = \beta_1$

$\overline{K_t} \rightarrow$ Transmitted propagation vector

$|\overline{K_t}| = \beta_2$

Since the oblique incidence case is usually apply to a lossless medium ($\sigma=$

hence The general expression for the E-field for arbitrary direction in the lossless medium is

$$E = E_0\, e^{-j\beta \cdot r}\, a_E \qquad a_E \rightarrow \text{E-field unit vector.}$$

$r \rightarrow$ position vector, $r = x a_x + y a_y + z a_z$.

$E_0 \rightarrow$ Maximum amplitude of E-field.

The E-field lines lies in the xz-plane which is the plane of incidence. In medium 1, both incident and reflected fields are exists which are expressed as,

Incident wave:

$$E_i = E_{i0}\, e^{-j\beta_1 \cdot r}\, a_{E_i}$$

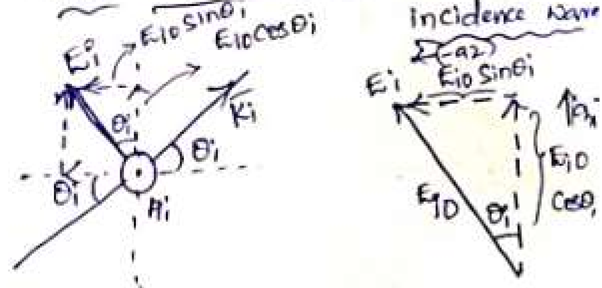
$$\therefore E_i = E_{i0}\left[\cos\theta_i a_x - \sin\theta_i a_z\right] * e^{-j\beta_1\left[\cos\theta_i a_z + \sin\theta_i a_x\right]\cdot\left[x a_x + y a_y + z a_z\right]}$$

$$\therefore \boxed{E_i = E_{i0}\left[\cos\theta_i a_x - \sin\theta_i a_z\right] e^{-j\beta_1\left[x\sin\theta_i + z\cos\theta_i\right]}} \rightarrow (1)$$

$$H_i = H_{i0}\, e^{-j\beta_1 \cdot r}\, a_y$$

$$\boxed{H_i = \frac{E_{i0}}{\eta_1}\, e^{-j\beta_1\left[x\sin\theta_i + z\cos\theta_i\right]} a_y} \rightarrow (2)$$

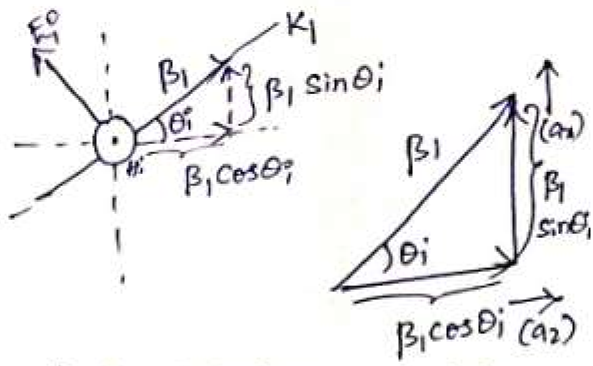
For E-field orientation in the incidence wave



$$\bar{E}_{i0} = E_{i0}\cos\theta_i a_x - E_{i0}\sin\theta_i a_z.$$

$$\bar{E}_{i0} = |E_{i0}|\, a_{E_i}$$

For wave orientation in the incident wave



$$\therefore \beta_1 = \beta_1\cos\theta_i a_z + \beta_1\sin\theta_i a_x.$$

Reflected wave:

$$E_r = E_{r0}\, e^{-j\beta_1 \cdot r}\, a_{E_r}.$$

$$= E_{r0}\left[\sin\theta_r a_z + \cos\theta_r a_x\right] e^{-j\left[-\beta_1\cos\theta_r a_z + \beta_1\sin\theta_r a_x\right]\cdot\left[x a_x + y a_y + z a_z\right]}$$

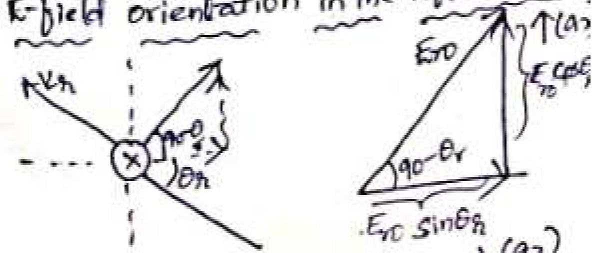
$$= E_{r0}\left[\cos\theta_r a_x + \sin\theta_r a_z\right] e^{-j\beta_1\left[\sin\theta_r a_x - \cos\theta_r a_z\right]\cdot\left[x a_x + y a_y + z a_z\right]}$$

$$\boxed{E_r = E_{r0}\left[\cos\theta_r a_x + \sin\theta_r a_z\right] e^{-j\beta_1\left[x\sin\theta_r - z\cos\theta_r\right]}} \rightarrow (3)$$

$$H_r = H_{r0}\, e^{-j\beta_1\cdot r}\,(-a_y)$$

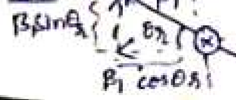
$$\boxed{H_r = -\frac{E_{r0}}{\eta_1}\, e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} a_y} \rightarrow (4)$$

E-field orientation in the reflected wave



$$\bar{E}_{r0} = E_{r0}\sin\theta_r a_z + E_{r0}\cos\theta_r a_x$$

Wave :-



$$\bar{\beta}_1 = -\beta_1\cos\theta_r a_z + \beta_1\sin\theta_r a_x$$

where $\eta_1 \rightarrow$ impedance of Medium 1 $= \sqrt{\frac{j\omega\mu_1}{j\omega\varepsilon_1}}$ Since $\sigma = 0$

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{\mu_0\mu_{r1}}{\varepsilon_0\varepsilon_{r1}}} = \eta_0\sqrt{\frac{\mu_{r1}}{\varepsilon_{r1}}}.$$

$$\beta_1 = \omega\sqrt{\mu_1\varepsilon_1} \longrightarrow (5)$$

The transmitted fields exist in medium 2 and are given by

$$E_t = E_{t0}(\cos\theta_t a_x - \sin\theta_t a_z)\, e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \longrightarrow (6)$$

$$\boxed{H_t = \frac{E_{t0}}{\eta_2} e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} a_y} \longrightarrow (7)$$

where $\beta_2 = \omega\sqrt{\mu_2\varepsilon_2} \longrightarrow (8)$

Since the tangential components of $\vec{E}$ and $\vec{H}$ are continuous at the boundary i.e. $z=0$ when $\theta_r = \theta_i$.

$$(E_{i0} + E_{r0})\cos\theta_i = E_{t0}\cos\theta_t \longrightarrow (9)$$

& $H_{i0} + H_{r0} = H_{t0} \Rightarrow \frac{1}{\eta_1}[E_{i0} - E_{r0}]^{\cos\theta t} = \frac{1}{\eta_2}E_{t0} \longrightarrow (10)$

from eqn (10), $E_{t0} = \frac{\eta_2}{\eta_1}[E_{i0} - E_{r0}] \longrightarrow (11)$

from (9) & (11). $(E_{i0} + E_{r0})\cos\theta_i = \frac{\eta_2}{\eta_1}[E_{i0} - E_{r0}]\cos\theta_t$

$$\Rightarrow \eta_1 E_{i0}\cos\theta_i + \eta_1 E_{r0}\cos\theta_i = \eta_2 E_{i0}\cos\theta_t - \eta_2 E_{r0}\cos\theta_t$$

$$\Rightarrow \eta_1 E_{r0}\cos\theta_i + \eta_2 E_{r0}\cos\theta_t = \eta_2 E_{i0}\cos\theta_t - \eta_1 E_{i0}\cos\theta_i$$

$$\Rightarrow E_{r0}[\eta_1\cos\theta_i + \eta_2\cos\theta_t] = [\eta_2\cos\theta_t - \eta_1\cos\theta_i]E_{i0}$$

$E_{1\tan} = E_{2\tan}$

E-field having the component parallel to the boundary is tangential component and the component perpendicular to the boundary is the normal component. Hence a_x is the tangential and a_z is the normal component.

$\therefore E_{i0}\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + E_{r0}\cos\theta_r e^{-j\beta_1 x\sin\theta_r} = E_{t0}\cos\theta_t e^{-j\beta_2 x\sin\theta_t}$

Since $\theta_i = \theta_r$ and

$\frac{\sin\theta_t}{\sin\theta_i} = \frac{\beta_1}{\beta_2} = \frac{\sqrt{\mu_1\varepsilon_1}}{\sqrt{\mu_2\varepsilon_2}}$

$\therefore \beta_2\sin\theta_t = \beta_1\sin\theta_i$

$\therefore (E_{i0} + E_{r0})\cos\theta_i = E_{t0}\cos\theta_t$

$$\boxed{\frac{E_{r0}}{E_{i0}} = \Gamma_{/\!/} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}} \longrightarrow (12)$$

$\rightarrow$ Reflection Coefficient in the case of parallel polarization

Since $1 + \Gamma_{/\!/} = T_{/\!/} \Rightarrow T_{/\!/} = 1 + \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$

$$\boxed{T_{/\!/} = \frac{2\eta_2\cos\theta_t}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}} \longrightarrow (13)$$

$\rightarrow$ Transmission coefficient in the case of perpendicular polarization.

from Snell's law

Since $\dfrac{\sin\theta_t}{\sin\theta_i} = \sqrt{\dfrac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}} = \dfrac{n_1}{n_2} = \dfrac{\beta_1}{\beta_2} \rightarrow (14)$ $n_1, n_2 \rightarrow$ refractive index

Equations (12) & (13) are called 'Fresnel's Equations'. Since θ_t and θ_i are related according to Snell's law. hence these equations can be written interms of θ_i by substituting

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-\left(\frac{\mu_2}{\mu_1}\right)^2 \sin^2\theta_i}$$

from equations (12) & (13). It can be shown that

$$\boxed{1+\Gamma_{||} = T_{||}\left(\frac{\cos\theta_t}{\cos\theta_i}\right)} \longrightarrow (15)$$

from equation (12), it is evident that it is possible that $\Gamma_{||}=0$ because the numerator is the difference of two terms.

The angle of incidence (θ_i) at which there is no reflection at the interface $[E_{r0}=0]$ is called the 'Brewster angle' ($\theta_{B||}$).

The Brewster angle is also known as the 'polarizing angle' because an arbitrary polarized incident wave will be reflected with only the component of E perpendicular to the plane of incidence.

The Brewster angle is obtained by setting $\theta_i = \theta_{B||}$ when $\Gamma_{||}=0$ in equation (12).

$$n_2\cos\theta_t = n_1\cos\theta_{B||}$$

Squaring on both sides,

$$n_2^2[1-\sin^2\theta_t] = n_2^2\cos^2\theta_t = n_1^2\cos^2\theta_{B||} = n_1^2\left[1-\sin^2\theta_{B||}\right]$$

By Introducing eqn (14),

$$\boxed{\sin^2\theta_{B||} = \frac{1-(\mu_2\varepsilon_1/\mu_1\varepsilon_2)}{1-(\varepsilon_1/\varepsilon_2)^2}} \longrightarrow (16)$$

When the dielectric mediums are not only lossless but nonmagnetic as well, that is $\mu_1=\mu_2=\mu_0$. Then equation (16) gives

$$\sin^2\theta_{B||} = \frac{1}{1+\varepsilon_1/\varepsilon_2} \Rightarrow \sin\theta_{B||} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2}} \longrightarrow (17)$$

$$\tan\theta_B = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{\eta_2}{\eta_1} \longrightarrow (18)$$

Proof of Eqn (16)

$$\sin^2\theta_{B\parallel} = \frac{1-\mu_2\varepsilon_1/\mu_1\varepsilon_2}{1-(\varepsilon_1/\varepsilon_2)^2}$$

Proof: — At Brewster angle. $\theta_i = \theta_{B\parallel} \Rightarrow$

$$\eta_2 \cos\theta_t = \eta_1 \cos\theta_{B\parallel}$$

$$\Rightarrow \eta_2^2 \cos^2\theta_t = \eta_1^2 \cos^2\theta_{B\parallel}$$

$$\Rightarrow \eta_2^2\left[1-\sin^2\theta_t\right] = \eta_1^2\left[1-\sin^2\theta_{B\parallel}\right]$$

$\because$ from Snell's law, $\sin^2\theta_t = \frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_{B\parallel}$

$$\Rightarrow \eta_2^2\left[1-\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_{B\parallel}\right] = \eta_1^2\left[1-\sin^2\theta_{B\parallel}\right]$$

$$\Rightarrow \eta_2^2 - \eta_2^2\left[\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\right]\sin^2\theta_{B\parallel} = \eta_1^2 - \eta_1^2\sin^2\theta_{B\parallel}$$

$$\Rightarrow \sin^2\theta_{B\parallel}\left[\eta_1^2 - \eta_2^2\left(\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\right)\right] = \eta_1^2 - \eta_2^2$$

where $\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} \rightarrow$ Impedance of Medium 1, $\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} \rightarrow$ Impedance of Medium 2

$$\eta_1^2 = \frac{\mu_1}{\varepsilon_1}, \quad \eta_2^2 = \frac{\mu_2}{\varepsilon_2}$$

$$\sin^2\theta_{B\parallel}\left[\frac{\mu_1}{\varepsilon_1} - \frac{\cancel{\mu_2}}{\varepsilon_2}\left[\frac{\mu_1\varepsilon_1}{\cancel{\mu_2}\varepsilon_2}\right]\right] = \frac{\mu_1}{\varepsilon_1} - \frac{\mu_2}{\varepsilon_2} \Rightarrow \sin^2\theta_{B\parallel}\left[\frac{\mu_1}{\varepsilon_1} - \frac{\mu_1\varepsilon_1}{\varepsilon_2^2}\right] = \frac{\mu_1}{\varepsilon_1} - \frac{\mu_2}{\varepsilon_2}$$

$$\Rightarrow \sin^2\theta_{B\parallel} = \frac{(\mu_1\varepsilon_2 - \mu_2\varepsilon_1)/\varepsilon_1\varepsilon_2}{\mu_1\left[1/\varepsilon_1 - \frac{\varepsilon_1}{\varepsilon_2^2}\right]} = \frac{\mu_1\varepsilon_2 - \varepsilon_1\mu_2/\varepsilon_1\varepsilon_2}{\mu\varepsilon_2^2 - \mu\varepsilon_1^2/\varepsilon_2^2\varepsilon_1} = \frac{\mu_1\varepsilon_2 - \varepsilon_1\mu_2 * \varepsilon_2}{\mu\left[\varepsilon_2^2-\varepsilon_1^2\right]}$$

$$\Rightarrow \sin^2\theta_{B\parallel} = \frac{\mu_1\varepsilon_2 - \varepsilon_1\mu_2}{\varepsilon_2^2-\varepsilon_1^2} * \frac{\varepsilon_2}{\mu_1} = \frac{\cancel{\mu_1}\varepsilon_2\left(1-\frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\right) * \varepsilon_2}{\cancel{\mu_1}\left[\varepsilon_2^2-\varepsilon_1^2\right]} = \frac{\cancel{\varepsilon_2^2}\left[1-\frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\right]}{\cancel{\varepsilon_2^2}\left[1-\frac{\varepsilon_1^2}{\varepsilon_2^2}\right]}$$

$$\boxed{\sin^2\theta_{B\parallel} = \frac{\left[1-\frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}\right]}{1-\varepsilon_1^2/\varepsilon_2^2} = \frac{1-\frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}}{1-(\varepsilon_1/\varepsilon_2)^2}}$$

7 (ii) Perpendicular polarization :-

When the E-field is perpendicular to the plane of incidence [xz-plane], we have perpendicular polarization. Here in this case, H-field is parallel to the plane of incidence. The incident and reflected fields in medium 1 are given by:

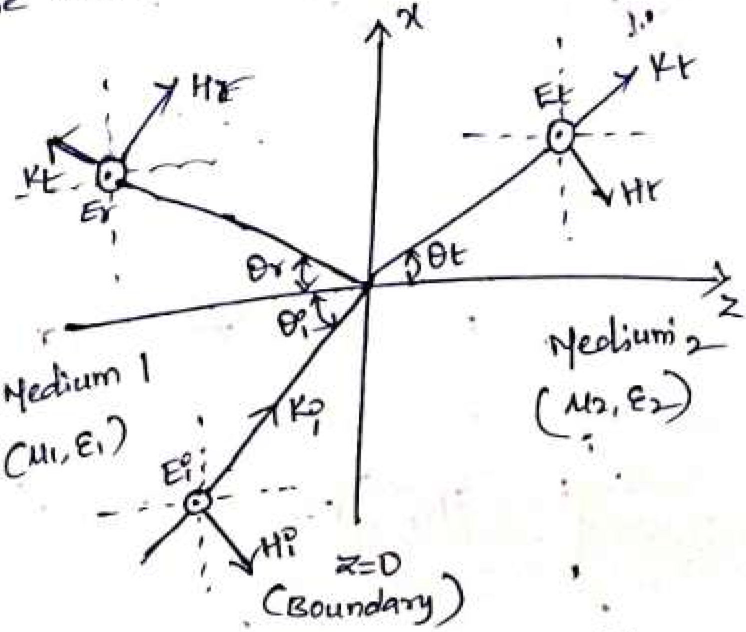
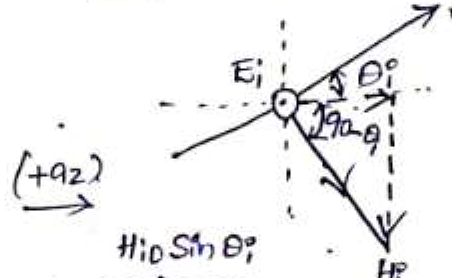
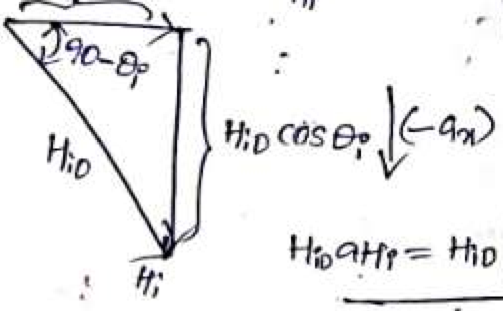


fig:- oblique incidence with E perpendicular to the plane of incidence

Incident wave :-

(i) H-field orientation:-

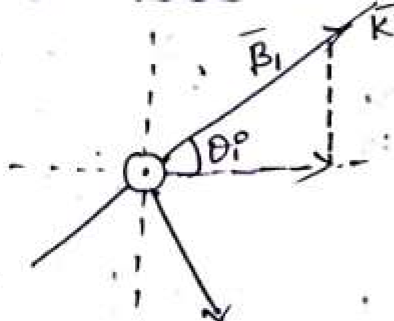


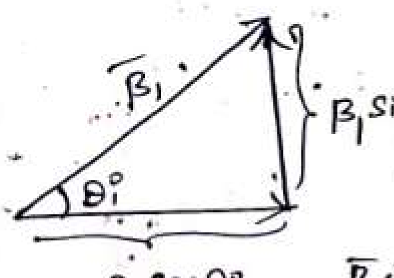


$$H_{i0}a_{Hi} = H_{i0}\left[\sin\theta_i a_z - \cos\theta_i a_x\right]$$

$$\bar{r} = x a_x + y a_y + z a_z$$

(ii) Wave orientation:-





$\beta_1 \cos\theta_i$ $(+a_z)$

$$\bar{\beta}_1 = \beta_1 \sin\theta_i a_x + \beta_1 \cos\theta_i a_z$$

$$\bar{\beta}_1 \cdot \bar{r} = \beta_1 x \sin\theta_i + \beta_1 z \cos\theta_i$$

for the arbitrary orientation of H-field and wave; $H_i = H_{i0} e^{-j\bar{\beta}_i \cdot \bar{r}} a_{Hi}$

$$\therefore H_i = H_{i0}\left[+\sin\theta_i a_z - \cos\theta_i a_x\right] e^{-j\left[\beta_1 x \sin\theta_i + \beta_1 z \cos\theta_i\right]}$$

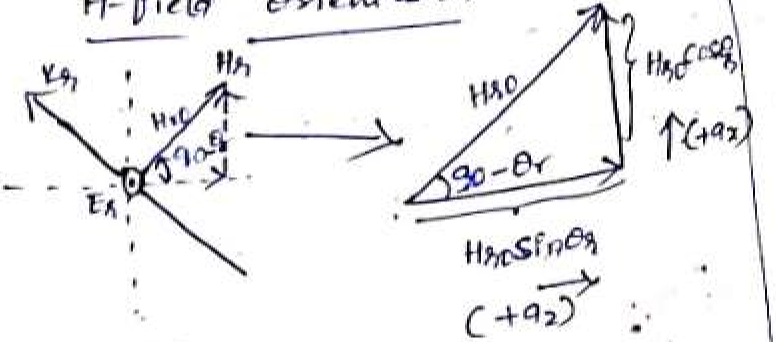
$$H_i = H_{i0}\left[-\cos\theta_i a_x + \sin\theta_i a_z\right] e^{-j\beta_1\left[x\sin\theta_i + z\cos\theta_i\right]} \longrightarrow (1)$$

where $H_{i0} = \dfrac{E_{i0}}{\eta_1}$

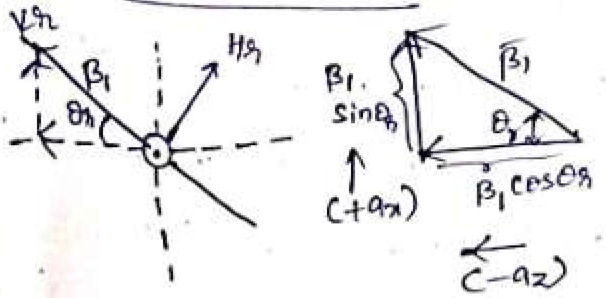
$$\boxed{E_i = E_{i0}\, e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} a_y} \longrightarrow (2)$$

Reflected Wave :-

H-field orientation



Wave orientation



$$\therefore \bar{\beta_1} = -\beta_1 \cos\theta_r a_z + \beta_1 \sin\theta_r a_x$$

$$\bar{\beta_1}\cdot\bar{r} = \beta_1\left[-z\cos\theta_r + x\sin\theta_r\right]$$

$$H_{r0}\, a_{rH} = H_{r0}\left[\sin\theta_r a_z + \cos\theta_r a_x\right]$$

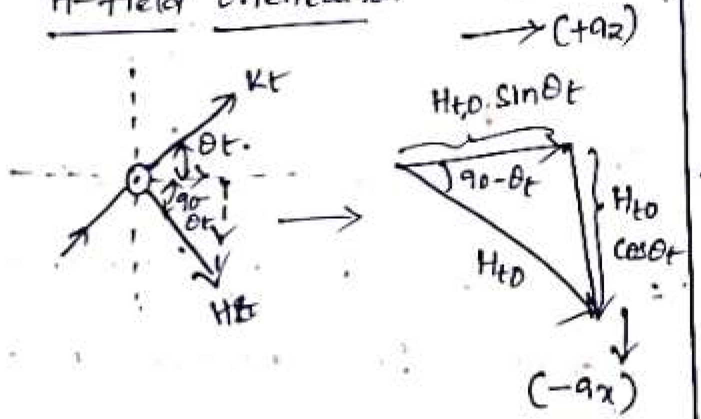
$$\therefore \boxed{H_r = H_{r0}\left[\sin\theta_r a_z + \cos\theta_r a_x\right] e^{-j\beta_1\left[x\sin\theta_r - z\cos\theta_r\right]}} \rightarrow (3)$$

$$\boxed{E_r = E_{r0}\, e^{-j\beta_1\left[x\sin\theta_r - z\cos\theta_r\right]} a_y} \rightarrow (4)$$

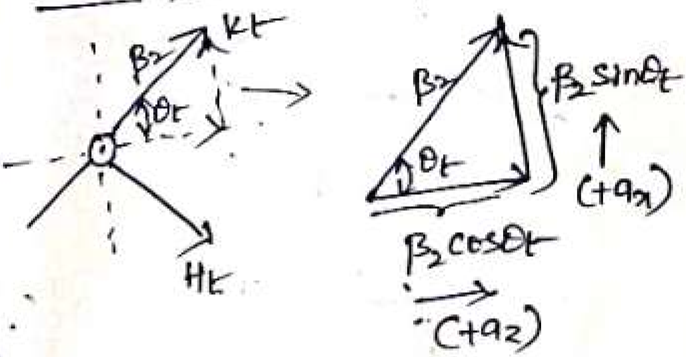
where $H_{r0} = \dfrac{E_{r0}}{\eta_1}$

Transmitted wave:

H-field orientation



Wave orientation



$$\bar{\beta_2} = \beta_2\sin\theta_t a_x + \beta_2\cos\theta_t a_z$$

$$\bar{\beta_2}\cdot\bar{r} = \beta_2\left[x\sin\theta_t + z\cos\theta_t\right]$$

$$H_{t0}\, a_{H_t} = H_{t0}\left[-\cos\theta_t a_x + \sin\theta_t a_z\right]$$

$$\therefore \boxed{H_t = H_{t0}\left[-\cos\theta_t a_x + \sin\theta_t a_z\right] e^{-j\beta_2\left[x\sin\theta_t + z\cos\theta_t\right]}} \rightarrow (5)$$

$$\therefore \boxed{E_t = E_{t0}\, e^{-j\beta_2\left[x\sin\theta_t + z\cos\theta_t\right]} a_y} \rightarrow (6)$$

where $H_{t0} = \dfrac{E_{t0}}{\eta_2}$

From the Boundary conditions, the tangential Components of E and H are continuous at the boundary separating Dielectric to Dielectric interface when $\theta_i = \theta_r$. Hence

$$E_{1tan} = E_{2tan}$$

$$E_i\Big|_{z=0} + E_r\Big|_{z=0} = E_t\Big|_{z=0}$$

$$E_{i0}\, e^{-j\beta_1 x \sin\theta_i} + E_{r0}\, e^{-j\beta_1 x \sin\theta_r} = E_{t0}\, e^{-j\beta_2 x \sin\theta_t}$$

Since $\theta_i = \theta_r$

$$\Rightarrow (E_{i0} + E_{r0})\, e^{-j\beta_1 x \sin\theta_i} = E_{t0}\, e^{-j\beta_2 x \sin\theta_t}$$

from Snell's law. $\frac{\sin\theta_t}{\sin\theta_i} = \frac{\beta_1}{\beta_2} \Rightarrow \beta_2 \sin\theta_t = \beta_1 \sin\theta_i$

$$\therefore (E_{i0} + E_{r0})\, e^{-j\beta_1 \sin\theta_i x} = E_{t0}\, e^{-j\cancel{\beta_1 \sin\theta_i x}}$$

$$\Rightarrow \boxed{E_{i0} + E_{r0} = E_{t0}} \longrightarrow (7)$$

$$H_{1tan} = H_{2tan}$$

$$H_i\Big|_{z=0} + H_r\Big|_{z=0} = H_t\Big|_{z=0} \Rightarrow H_{i0}[-\cos\theta_i]\, e^{-j\beta_1 x \sin\theta_i} + H_{r0}[\cos\theta_r]\, e^{-j\beta_1 x \sin\theta_r}$$

$$= H_{t0}[-\cos\theta_t]\, e^{-j\beta_2 x \sin\theta_t}$$

$$\Rightarrow H_{i0} \cos\theta_i\, e^{-j\beta_1 x \sin\theta_i} - H_{r0} \cos\theta_r\, e^{-j\beta_1 x \sin\theta_r} = H_{t0} \cos\theta_t\, e^{-j\beta_2 x \sin\theta_t}$$

When $\theta_i = \theta_r \Rightarrow$

$$[H_{i0} \cos\theta_i - H_{r0} \cos\theta_i]\, e^{-j\beta_1 x \sin\theta_i} = H_{t0} \cos\theta_t\, e^{-j\beta_2 x \sin\theta_t}$$

By Snell's law. $\frac{\sin\theta_t}{\sin\theta_i} = \frac{\beta_1}{\beta_2} \Rightarrow \beta_2 \sin\theta_t = \beta_1 \sin\theta_i$

$$\therefore (H_{i0} - H_{r0}) \cos\theta_i * e^{-j\beta_1 x \sin\theta_i} = H_{t0} \cos\theta_t\, e^{-j\beta_1 x \sin\theta_i}$$

$$\Rightarrow (H_{i0} - H_{r0}) \cos\theta_i = H_{t0} \cos\theta_t \Rightarrow \left(\frac{E_{i0}}{\eta_1} - \frac{E_{r0}}{\eta_1}\right) \cos\theta_i = \frac{E_{t0}}{\eta_2} \cos\theta_t$$

$$\Rightarrow \frac{1}{\eta_1}[E_{i0} - E_{r0}]\cos\theta_i = \frac{1}{\eta_2} E_{t0}\cos\theta_t \longrightarrow (8)$$

from (7) & (8),

$$\Rightarrow \frac{1}{\eta_1}[E_{i0} - E_{r0}]\cos\theta_i = \frac{1}{\eta_2}[E_{i0} + E_{r0}]\cos\theta_t$$

$$\Rightarrow \frac{E_{r0}}{\eta_1}\cos\theta_i + \frac{E_{r0}}{\eta_2}\cos\theta_t = \frac{E_{i0}}{\eta_1}\cos\theta_i - \frac{E_{i0}}{\eta_2}\cos\theta_t$$

$$\Rightarrow E_{r0}\left[\frac{\eta_2\cos\theta_i + \eta_1\cos\theta_t}{\cancel{\eta_1\eta_2}}\right] = E_{i0}\left[\frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\cancel{\eta_1\eta_2}}\right]$$

$$\Rightarrow \boxed{\Gamma_\perp = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}} \longrightarrow (9)$$

Since $T_\perp = 1 + \Gamma_\perp$

$$\therefore \boxed{T_\perp = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}} \longrightarrow (10)$$

Equations (9) & (10) are "fresnel's Equations" for perpendicular polarization.

→ Brewster angle:-

The angle of incidence at which there is no reflected wave at the boundary is called "Brewster Angle" ($\theta_{B\perp}$).

At $\theta_i = \theta_{B\perp} \Rightarrow \Gamma_\perp = 0 \Rightarrow \boxed{\eta_2\cos\theta_i = \eta_1\cos\theta_t} \rightarrow (11)$

or $\quad \eta_2^2(1-\sin^2\theta_{B\perp}) = \eta_1^2(1-\sin^2\theta_t)$

by introducing snell's law, $\sin^2\theta_t = \frac{\mu_1\epsilon_1}{\mu_2\epsilon_2}\sin^2\theta_{B\perp}$

finally $\Rightarrow$ $$\boxed{\sin^2\theta_{B\perp} = \frac{1-\mu_1\epsilon_2/\mu_2\epsilon_1}{1-(\mu_1/\mu_2)^2}} \longrightarrow (12)$$

for nonmagnetic media ($\mu_1 = \mu_2 = \mu_0$), $\sin^2\theta_{B\perp} \to \infty$ in eqn (12), so $\theta_{B\perp}$ does not exist because the sine of an angle is never greater than unity. Also if $\mu_1 \neq \mu_2$ and $\varepsilon_1 = \varepsilon_2$, eqn (12) reduces to

$$\boxed{\sin\theta_{B\perp} = \sqrt{\frac{\mu_2}{\mu_1+\mu_2}}} \longrightarrow (13)$$

or

$$\boxed{\tan\theta_{B\perp} = \sqrt{\frac{\mu_2}{\mu_1}}} \longrightarrow (14)$$

Critical Angle (θ_c): — The angle of incidence at which the transmitted wave propagates at 90° to the normal is called 'critical angle'.

If the wave propagates across an interface such that the angle of refraction is larger than the angle of incidence, an increase in angle of incidence leads to an angle at which the transmitted wave propagates at 90° to the normal. This angle is called a "critical angle".

→ Any increase in the angle of incidence results in total reflection of the incident wave since what would have been the transmitted wave in medium 2 is transmitted into material/medium 1. This condition occurs in lossless dielectrics if $\varepsilon_1 > \varepsilon_2$.

→ This phenomenon exists in either $\perp$ or $\parallel$ polarization.

→ The total reflection occurs when the reflection coefficient is equal to unity.

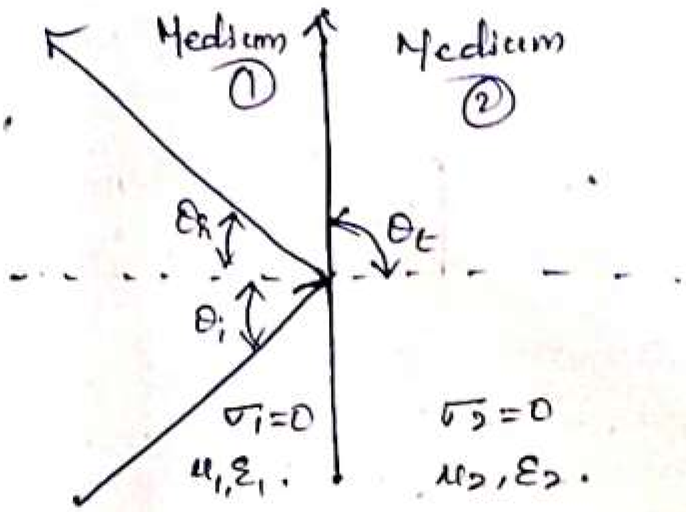
$$\Gamma_\perp = 1, \quad \Gamma_\parallel = -1 \quad \text{at } \theta_t = 90^\circ$$

Since $\sin\theta_t = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}}\sin\theta_i$

Substituting $\theta_t = 90^\circ$ gives the critical angle

$$\boxed{\sin\theta_c = \sqrt{\frac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}}} \text{ for } \mu_2\varepsilon_2 \le \mu_1\varepsilon_1$$

The condition $\mu_2\varepsilon_2 \le \mu_1\varepsilon_1$ is necessary otherwise, $\sin\theta_c$ would be larger than 1.



$$\boxed{\varepsilon_1 > \varepsilon_2}$$

Poynting Vector :-

Energy can be transported from one point (transmitting position) to another point (receiving position) by means of EM waves. The rate of such energy transportation can be obtained from Maxwell's equations:

$$\nabla\times E = -\mu\frac{\partial H}{\partial t} \longrightarrow (1)$$

$$\nabla\times H = \sigma E + \varepsilon\frac{\partial E}{\partial t} \longrightarrow (2)$$

Dotting both sides of eqn (2) with E gives

$$E\cdot(\nabla\times H) = \sigma E^2 + E\cdot\varepsilon\frac{\partial E}{\partial t} \longrightarrow (3)$$

But for any vector fields 'A' and 'B',

$$\nabla\cdot(A\times B) = B\cdot(\nabla\times A) - A\cdot(\nabla\times B)$$

Applying this vector identity to eqn (3) by letting $A = H$ & $B = E$, gives

$$H\cdot(\nabla\times E) + \nabla\cdot(H\times E) = \sigma E^2 + E\cdot\varepsilon\frac{\partial E}{\partial t} \longrightarrow (5)$$

from eqn (1), $H\cdot(\nabla\times E) = H\cdot\left(-\mu\frac{\partial H}{\partial t}\right) = -\frac{\mu}{2}\frac{\partial}{\partial t}(H\cdot H) \longrightarrow (6)$

(17) and hence eqn (5) becomes

$$-\frac{\mu}{2}\frac{\partial H^2}{\partial t} - \nabla\cdot(E\times H) = \sigma E^2 + \frac{1}{2}\epsilon\frac{\partial E^2}{\partial t}$$

Rearranging terms and taking the volume integral on both sides,

$$\int_V \nabla\cdot(E\times H)\,dv = -\frac{\partial}{\partial t}\int_V\left[\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right]dv - \int_V \sigma E^2\,dv \quad \longrightarrow (7)$$

Applying divergence theorem to the left-hand side gives

$$\underbrace{\oint_S (E\times H)\cdot ds}_{} = \underbrace{-\frac{\partial}{\partial t}\int_V\left[\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right]dv}_{} - \underbrace{\int_V \sigma E^2\,dv}_{} \quad \longrightarrow (8)$$

Total power leaving the volume = Rate of decrease in energy stored in electric and magnetic fields − Ohmic power dissipated. → (9)

Equation (8) is referred to as "poynting Theorem".

The second term in the eqn (9) is due to fact that the medium is conducting ($\sigma \neq 0$).

The quantity $E\times H$ on the left-hand side of eqn (8) is known as the "poynting vector" $\mathcal{P}$ in watts per square meter (W/m^2), that is

$$\boxed{\mathcal{P} = E\times H} \quad \longrightarrow (10)$$

It represents the instantaneous power density vector associated with the EM field at a given point. The integration of the poynting vector over any closed surface gives the net power flowing out of that surface.

*> Poynting theorem states that the net power flowing out of a given volume 'V' is equal to the time rate of decrease in the energy stored within 'V' minus the conduction losses.

$\mathcal{P}$ is normal to both E and H and is therefore along the direction of wave propagation a_k for uniform plane waves.

Thus, $\boxed{a_k = a_E \times a_H}$

Here '$\mathcal{P}$' points along 'a_k'.

If we assume that

$$E(z,t) = E_0 e^{-\alpha z} \cos(\omega t - \beta z) a_x$$

then

$$H(z,t) = \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta) a_y$$

and

$$\mathcal{P}(z,t) = \frac{E_0^2}{|\eta|} e^{-2\alpha z} \cos(\omega t - \beta z) \cos(\omega t - \beta z - \theta_\eta) a_z$$

$$= \frac{E_0^2}{2|\eta|} e^{-2\alpha z} \left[\cos\theta_\eta + \cos(2\omega t - 2\beta z - \theta_\eta) \right] a_z \longrightarrow (11)$$

The average poynting vector $\mathcal{P}_{ave}(z)$ (in W/m^2) is of ~~more p~~ more practical value than the instantaneous poynting vector $\mathcal{P}(z,t)$ over the period $T = 2\pi/\omega$; that is

$$\boxed{\mathcal{P}_{ave}(z) = \frac{1}{T}\int_0^T \mathcal{P}(z,t)\,dt} \longrightarrow (12)$$

Also,

$$\mathcal{P}_{ave}(z) = \frac{1}{2} Re\left(E_s \times H_s^* \right) \longrightarrow (13)$$

By substituting eqn (11) into eqn (12), we obtain

$$\boxed{\mathcal{P}_{ave}(z) = \frac{E_0^2}{2|\eta|} e^{-2\alpha z} \cos\theta_\eta \, a_z.} \longrightarrow (14)$$

The total time-average power crossing a given surface 'S' is given by

$$\boxed{P_{ave} = \int_S \mathcal{P}_{ave} \cdot ds} = P_{ave} \rightarrow \text{Total time average power through a surface in watts [Scalar]}$$

$\mathcal{P} \rightarrow$ Poynting vector. $\mathcal{P}_{ave}(x,y,z,t) \rightarrow$ poynting vector is time varying. $\mathcal{P}_{ave}(x,y,z) \rightarrow$ time average of Poynting vector.

TTL VII (1)

NATL Unit-IV
Transmission Lines-I

→ Transmission Lines Introduction:

→ Energy can be transmitted either by the ~~rad~~ radiation of free Electromagnetic waves as in the radio or it can be ~~constructed~~ constrained to move or carried in various conductor arrangement known as "Transmission Line". Thus a transmission line is a conductive method of guiding electrical energy from one place to another.

→ In communication, these lines are used as a link between an anten and a transmitter or a receiver.

Transmission Lines Types :—

→ Basically there are four types of Transmission Lines:

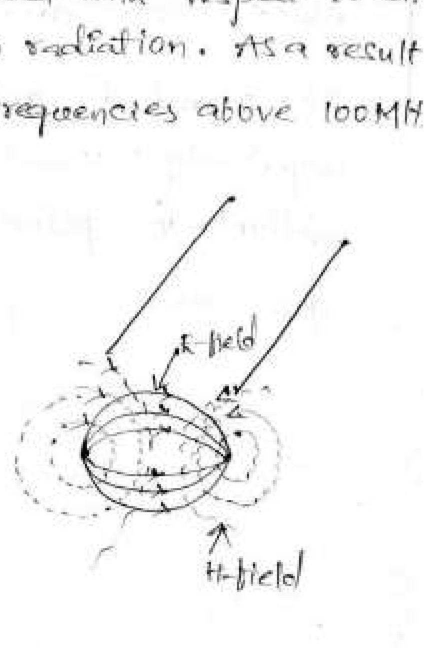
(1) parallel wire type
(2) coaxial type
(3) waveguides
(4) optical fibers

Parallel wire Type :—

is a common form of transmission line, also known as open wi line because of its construction.

They are commonly employed as Telephone lines, Telegraphy lines and power lines. These lines are easy to construct and are cheaper since insulation between line conductors is normally air, the dielectric loss is extremely small. open wire lines is balanced with respect to ear However, there is significant energy loss due to radiation. As a result of which these lines becomes unsuitable for frequencies above 100MH

→ Electrical Energy propagating through these lines set up electric fields between line conductors. These fields are at right angles to each other and to the direction of propagation and this type of Energy transmission is commonly known as "Transverse Electromagnetic Mode of propagation".

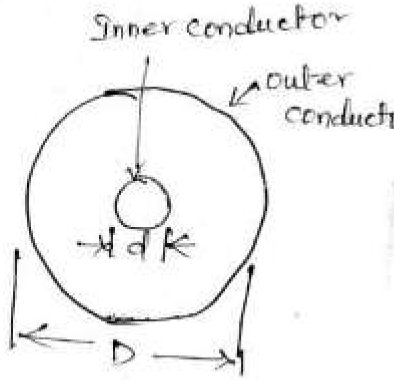


Coaxial Type :-

In this one conductor is a hollow tube, the second conductor being located inside and coaxial with the tube. The dielectric may be solid or gaseous.

Inorder to avoid severe radiation losses taken place in open wire lines at frequencies beyond 100MHz, a closed field configuration is employed in coaxial cable by surrounding the inner conductor with an outer cylindrical hollow conductor.

→ The important advantage of coaxial is that electric and magnetic fields remain confined within the outer conductor and cannot leak into free space. As a consequence of which radiation is totally eliminated. Apart from this, the outer conductor also provides a highly effective electromagnetic shielding against external electromagnetic signals usually have a continuous dielectric.



→ coaxial cables are extensively used in the frequency range extending upto 1GHz and at frequencies beyond 1GHz these cables become unusable. These cables are quite costlier as compared to open wire lines. Losses in the dielectric increases as the signal frequency is increased. These losses becomes excessive at frequencies above 1GHz and which, these cannot be used.

→ If 'D' and 'd' are the diameters of the outer and inner conductors respectively; 'μ' and 'ε' are the permeability and permittivity of insulating medium, the primary constants of a coaxial cable are then approximately given as

$$L = \frac{\mu}{2\pi} \log(D/d) \ H/m \qquad C = 2\pi\epsilon / \log(D/d) \ F/m$$

TTL
VII (2)

Waveguides :— A transmission line consisting of a suitable shaped hollow conductor, which may be fitted with a dielectric material and is used to guide electromagnetic waves of UHF propagated along its length, is called a "Waveguide". The transmitted wave is reflected back by the internal walls of waveguide and the resulting distributi[on] associated with the wave causes the transmission mode.

→ The main types of configuration are :

(a) TE Wave (Transverse Electric Wave)

(b) TM Wave (Transverse Magnetic Wave)

(c) TEM Wave (Transverse Electro-Magnetic Wave)

→ The TEM mode is one of the most commonly excited mode in coaxial cables. It cannot be propagated in a waveguide.

→ A waveguide in which no reflected wave occurs at any of the transverse section is called a 'Matched Waveguide'.

optical fibers :—

Because of Superior transmission quality, higher information carrying capacity, light weight and smaller size, reduced cost and higher security optical fibers are mostly used transmission lines over wired transmission lines.

→ Because of optical fibers low attenuation signals can be transmit[ted] for a considerable distance.

→ optical fibers has tremendous bandwidth or information carrying capacity. The system bandwidth can be increased by transmitting at more than one wavelength using multiple light sources. This is called 'Wave Division Multiplexing' which increases a information carrying capacity.

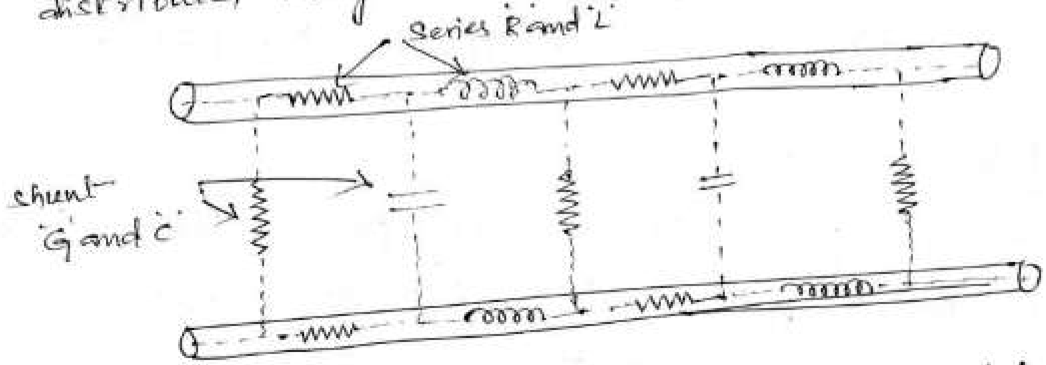
→ A transmission line basically consists of two or more parallel conductors used to connect a source to a load. The source may be a hydroelectric generator, a transmitter, or an oscillator; the load may be an antenna or an oscilloscope.

Transmission Line parameters:—

→ It is convenient to describe a transmission line in terms of it's line parameters, which are it's resistance per unit length R, Inductance per unit length L, conductance per unit length G and capacitance per unit length C.

→ It should be noted that.

1) The Line parameters R, L, G and C are not discrete or lumped. Rather, they are distributed i.e., the parameters are uniformly distributed along the entire length of the line.



(2) For each line, the conductors are characterized by $\sigma_c, \mu_c, \epsilon_c = \epsilon_0$ and the homogeneous dielectric separating the conductors is characterized by σ, μ, ϵ.

(3) $G \neq 1/R$; R is the ac resistance per unit length of the conductors comprising the line, and G is the conductance per unit length due to the dielectric medium separating the conductors.

(4) L is the external inductance per unit length i.e., $L = L_{ext}$ which are shown below. The effects of internal inductance $L_{in} (= R/\omega)$ are negligible at the high frequencies at which most communication systems operate.

(5) For each line,

$$LC = \mu\epsilon \quad \text{and} \quad \frac{G}{C} = \frac{\sigma}{\epsilon}$$

VII (3)

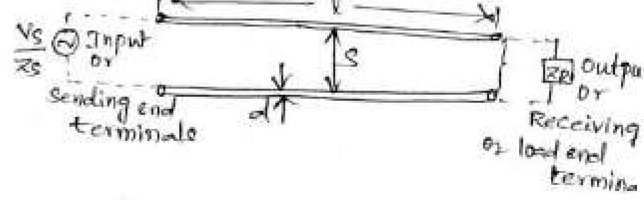
Distributed Line parameters at High frequencies:

Parameters	Coaxial Line	Two-wire Line	Planar Line
$R\,(\Omega/m)$	$\frac{1}{2\pi\delta\sigma_c}\left[\frac{1}{a}+\frac{1}{b}\right]$ $(\delta \ll a, c-b)$	$\frac{1}{\pi a\delta\sigma_c}$ $(\delta \ll a)$	$\frac{2}{w\delta\sigma_c}$ $(\delta \ll t)$
$L\,(H/m)$	$\frac{\mu}{2\pi}\ln(b/a)$	$\frac{\mu}{\pi}\cosh^{-1}(d/2a)$	$\frac{\mu d}{w}$
$G\,(S/m)$	$\frac{2\pi\sigma}{\ln(b/a)}$	$\frac{\pi\sigma}{\cosh^{-1}(d/2a)}$	$\frac{\sigma w}{d}$
$C\,(F/m)$	$\frac{2\pi\epsilon}{\ln(b/a)}$	$\frac{\pi\epsilon}{\cosh^{-1}(d/2a)}$	$\frac{\epsilon w}{d}$ $(w \gg d)$

Basic principles of Transmission Lines :-

→ Transmission lines are means to convey electrical signals or power between two points separated appreciably in distance. The simplest type of transmission line is a pair of parallel wires insulated from each other as shown in figure. It is also known as two-wire balanced transmission line.

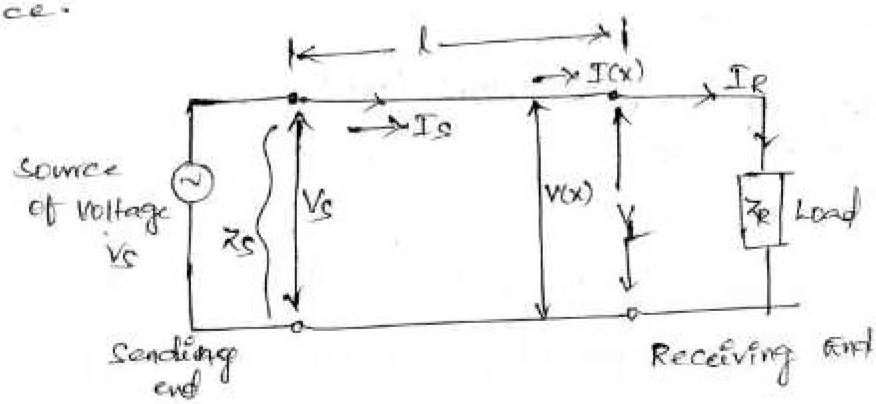
The diameter 'd', the spacing 's', the length 'l' are dimensional properties of the line.



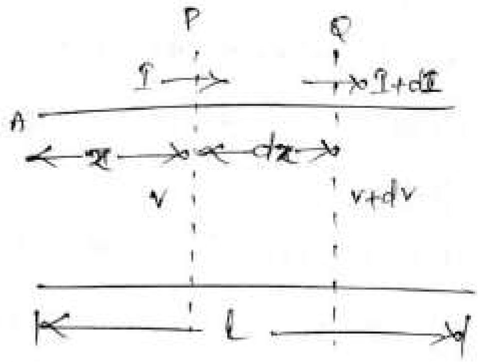
For the operation, the load impedance Z_R is connected at the Load terminals and a sinusoidal voltage V_S is applied to the input terminals of the line. Now the problem is to find out the current and voltage along the line (at any point) and at the load, if V_S and Z_R are given. Further the input impedance Z_S is also to be known. The input impedance Z_S is the ratio of input voltage V_S to the input current I_S and is easily calculated if V_S and I_S is determined.

→ Input impedance is the impedance that transmission line presents at its input terminals to the source of input voltage.

In a transmission line, as the input voltage V_s is applied at the input terminals, it takes a finite time in reaching to load end terminals. Therefore, a finite time is needed for the voltage and current to travel the length of the transmission line just as the electromagnetic wave has a finite velocity in space. Transmission line problem is analysed interms of its field rather interms of voltage and current. The transmission line can be regarded as a means of guiding the field so that it is confined near the line rather than spreading in space spherically. As the wave is confined, it doesnot suffer inverse-square law decrease of power-density with distance, unlike waves in free space.



Transmission Line Equations:—



Let the line be for length 'l' and primary constants of the line be 'R', 'L', 'C' and 'G' per KM. These constants does not vary with frequency.

Consider a short section of line PQ of length dx, at a distance 'x' from the sending end 'A' as shown in figure. By making 'dx' very small, the current may be considered constant for voltage calculations and voltage constant for current calculations.

At 'P', let the voltage be 'V' and current 'I'.

At 'Q', the voltage will be V+dV and current I+dI.

The series impedance of small section dx will be $(R+j\omega L)dx$. Similarly, the shunt admittance of small section dx will be $(G+j\omega C)dx$.

TTL VII (4)

Since dz is very small, the voltage drop from P to Q may be considered to be due to the current I flowing through the series impedance $(R+j\omega L)dz$. The decrease in current from P to Q may be considered to be due to the voltage V applied to the shunt admittance $(G+j\omega C)dz$.

→ potential difference between P and Q is due to the current flowing through the series impedance $(R+j\omega L)dz$. Thus

$$V-(V+dV) = I(R+j\omega L)dz$$

$$-dV = (R+j\omega L)I\,dz \Rightarrow \boxed{+\frac{dV}{dz} = -(R+j\omega L)I} \longrightarrow (1)$$

→ current difference between P and Q is due to the voltage applied to shunt admittance $(G+j\omega C)dz$. Thus,

$$I-(I+dI) = V(G+j\omega C)dz$$

$$-dI = (G+j\omega C)V\,dz \Rightarrow \boxed{\frac{dI}{dz} = -(G+j\omega C)V} \longrightarrow (2)$$

To eliminate I from eqn (1), differentiate eqn (1) with respect to z. Then

$$\frac{d^2V}{dz^2} = -(R+j\omega L)\frac{dI}{dz}$$

Since $\frac{dI}{dz} = -(G+j\omega C)V$

$$\therefore \boxed{\frac{d^2V}{dz^2} = (R+j\omega L)(G+j\omega C)V} \longrightarrow (3)$$

Similarly to eliminate V from eqn (2), differentiate eqn (2) with respect to z. Then

$$\frac{d^2I}{dz^2} = -(G+j\omega C)\frac{dV}{dz}$$

$$= -(G+j\omega C) * -(R+j\omega L)I$$

$$\boxed{\frac{d^2I}{dz^2} = (R+j\omega L)(G+j\omega C)I} \longrightarrow (4)$$

Therefore, $\frac{d^2V}{dz^2} = \gamma^2 V \longrightarrow (5)$ $\quad \frac{d^2I}{dz^2} = \gamma^2 I \longrightarrow (6)$

where $\boxed{\gamma^2 = (R+j\omega L)(G+j\omega C)}$, γ is a complex constant for a given frequency $\rightarrow (7)$

Equations ⑤ & ⑥ are referred to as 'Differential Equations of the Transmission Line'. These equations are standard linear differential equations with constant coefficients whose solutions are

***
$$\boxed{\begin{aligned} V &= Ae^{-\gamma z} + Be^{+\gamma z} \\ I &= Ce^{-\gamma z} + De^{+\gamma z} \end{aligned}}$$

$\longrightarrow$ ⑧

$\longrightarrow$ ⑨

where 'A' and 'B' are constants with the dimension of voltage while 'C' and 'D' are constants with the dimensions of current.

From eqn ⑦, propagation constant 'γ' is a complex quantity that may be represented as,

$$\gamma^2 = (R+j\omega L)(G+j\omega C)$$

Since $\gamma = \alpha + j\beta$

Therefore from eqns ⑧ & ⑨,

$$V = Ae^{-(\alpha+j\beta)z} + Be^{+(\alpha+j\beta)z}, \quad I = Ce^{-(\alpha+j\beta)z} + De^{+(\alpha+j\beta)z}$$

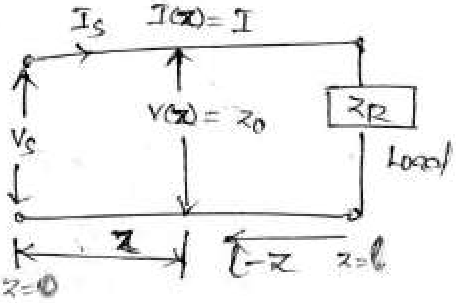
$$\boxed{V = \underbrace{Ae^{-\alpha z}e^{-j\beta z}}_{\text{I term}} + \underbrace{Be^{\alpha z}e^{j\beta z}}_{\text{II term}}, \quad I = \underbrace{Ce^{-\alpha z}e^{-j\beta z}}_{\text{I term}} + \underbrace{De^{\alpha z}\cdot e^{j\beta z}}_{\text{II-term}}}$$

$\longrightarrow$ 10(a) $\longrightarrow$ 10(b)

The first term in the above eqns of voltage and current show that there exists a voltage and current components travelling towards the load impedance. These are called "Incident waves" which decays exponentially in the positive direction of 'z', i.e., away from source and suffers a phase shift of 'β' radians per unit length of transmission line. Similarly, the secondary terms represents a wave similar to incident wave but travelling from the load end of the line towards source end. These are known as 'Reflected waves'.

TTL -VII (5)

General Solution of a Transmission Line Terminated with any Load Impedance (Z_R):—



Consider the figure in which load impedance or terminating impedance is Z_R ($\neq Z_0$).

Since load impedance is not equal to characteristic impedance Z_0, so the general equations of voltage and current may be used. Since we have,

$$V = Ae^{-\gamma z} + Be^{+\gamma z} \longrightarrow (1)$$

$$I = Ce^{-\gamma z} + De^{+\gamma z} \longrightarrow (2)$$

and we have $-\frac{dV}{dz} = I(R+j\omega L) \longrightarrow (3)$

Now differentiating eqn (1) with respect to z, we have

$$\frac{dV}{dz} = \left[A.e^{-\gamma z}(-\gamma) + Be^{+\gamma z}(\gamma)\right] \longrightarrow (4)$$

Sub eqn (4) in eqn (3), then

$$(Ae^{-\gamma z} - Be^{+\gamma z})\gamma = I(R+j\omega L)$$

$$\Rightarrow I = \frac{\gamma}{R+j\omega L}\left[Ae^{-\gamma z} - Be^{\gamma z}\right]$$

$$I = \frac{\sqrt{(R+j\omega L)(G+j\omega C)}}{(R+j\omega L)}\left[Ae^{-\gamma z} - Be^{\gamma z}\right]$$

$$\boxed{I = \sqrt{\frac{G+j\omega C}{R+j\omega L}}\,(Ae^{-\gamma z} - Be^{+\gamma z}) = \frac{1}{Z_0}\left[Ae^{-\gamma z} - Be^{\gamma z}\right]} \longrightarrow (5)$$

where $Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$ = characteristic impedance $\longrightarrow (6)$

since $\cosh\gamma z = \frac{e^{\gamma z} + e^{-\gamma z}}{2}$, $\sinh\gamma z = \frac{e^{\gamma z} - e^{-\gamma z}}{2}$

$\therefore \cosh\gamma z - \sinh\gamma z = \frac{1}{2}\left[\cancel{e^{\gamma z}} + e^{-\gamma z} - \cancel{e^{\gamma z}} + e^{-\gamma z}\right] = e^{-\gamma z} \longrightarrow (7)$

Similarly $\cosh\gamma z + \sinh\gamma z = \frac{1}{2}\left[e^{\gamma z} + \cancel{e^{-\gamma z}} + e^{\gamma z} - \cancel{e^{-\gamma z}}\right] = e^{\gamma z} \longrightarrow (8)$

By substituting eqns (7) & (8) in eqn (5) & (1), then we have

$$V = A[\cosh \gamma z - \sinh \gamma z] + B[\cosh \gamma z + \sinh \gamma z]$$

(or) $$\boxed{V = (A+B)\cosh \gamma z - (A-B)\sinh \gamma z} \longrightarrow (9)$$

and $$I = \frac{1}{Z_0}\left[A(\cosh \gamma z - \sinh \gamma z) - B(\cosh \gamma z + \sinh \gamma z)\right]$$

$$\boxed{I = \frac{1}{Z_0}\left[(A-B)\cosh \gamma z - (A+B)\sinh \gamma z\right]} \longrightarrow (10)$$

Now the condition at the input end is known i.e., at $z=0$, voltage is V_S and current is I_S then at $z=0$, $\cosh \gamma z = 1$ and $\sinh \gamma z = 0$ and also at $z=0$, $V = V_S$ and $I = I_S$. Hence from eqns (9) & (10),

$$V_S = A+B, \quad I_S = \frac{A-B}{Z_0} \longrightarrow (11)$$

Putting eqns (11) into eqn's (9) & (10), we get the complete general solution of voltage and current along the transmission line.

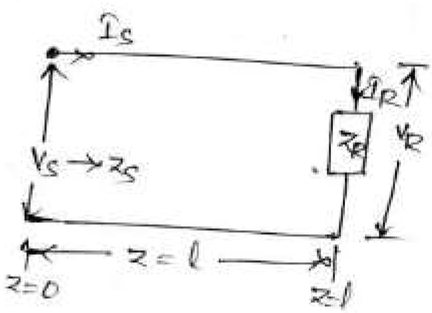
$$\therefore \boxed{\begin{aligned} V &= V_S \cosh \gamma z - I_S Z_0 \sinh \gamma z \\ I &= I_S \cosh \gamma z - \left(\frac{V_S}{Z_0}\right)\sinh \gamma z \end{aligned}}$$

and

These are complete general solutions of a transmission line for voltage and current at a point, distance 'z' from the sending end interms of sending end voltage (V_S) and current (I_S) and characteristic impedance Z_0.

Input Impedance of a Transmission Line Terminated with any Load Impedance (Z_R):—

Since the impedance offered by the line at its input terminals is to be calculated, so it will be convenient to use the equations that corresponding to sending end or input end as origin. A transmission line terminated with any load impedance Z_R at $z=l$ as shown in figure. The equations of voltage (V) and current (I) at a point distance 'z' from sending end are given by



U-III ⑥

$$V = V_S \cosh \gamma z - I_S Z_0 \sinh \gamma z \longrightarrow (1)$$

and $$I = I_S \cosh \gamma z - \frac{V_S}{Z_0} \sinh \gamma z \longrightarrow (2)$$

If the line is terminated at a length $z=l$, with any impedance Z_R, then voltage across terminated load impedance Z_R is given by

$$V = I_R Z_R \longrightarrow (3)$$

and the general solutions of the line terminated with any impedance Z_R at $z=l$ is given by eqns ① & ② . (Putting $V=V_R$ and $I=I_R$ and $z=l$) as

$$V_R = V_S \cosh \gamma l - I_S Z_0 \sinh \gamma l \longrightarrow (4)$$

$$I_R = I_S \cosh \gamma l - \frac{V_S}{Z_0} \sinh \gamma l \longrightarrow (5)$$

By putting these eqns ④ & ⑤ in eqn ③, we have

$$(V_S \cosh \gamma l - I_S Z_0 \sinh \gamma l) = Z_R \left[I_S \cosh \gamma l - \frac{V_S}{Z_0} \sinh \gamma l \right] \longrightarrow ⑥$$

Multiplying the above eqn both sides by Z_0,

$$Z_0 [V_S \cosh \gamma l - I_S Z_0 \sinh \gamma l] = Z_R [I_S Z_0 \cosh \gamma l - V_S \sinh \gamma l]$$

$$V_S [Z_0 \cosh \gamma l + Z_R \sinh \gamma l] = I_S Z_0 [Z_R \cosh \gamma l + Z_0 \sinh \gamma l]$$

$$\Rightarrow \boxed{\frac{V_S}{I_S} = Z_0 \frac{[Z_R \cosh \gamma l + Z_0 \sinh \gamma l]}{[Z_0 \cosh \gamma l + Z_R \sinh \gamma l]}}$$

Hence the sending end impedance or input impedance of the line as viewed by the source is given by

$$Z_S = \frac{V_S}{I_S} = Z_0 \left[\frac{Z_R \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_R \sinh \gamma l} \right]$$

Thus the sending end impedance Z_S (or input impedance of the line)

***
$$\boxed{Z_S = Z_0 \left[\frac{Z_R \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_R \sinh \gamma l} \right] = Z_{in}}$$

→ If the line is terminated by it's characteristic impedance Z_0 (i.e. $Z_R = Z_0$), is then characteristic impedance of the line,

$$Z_S = Z_0 \left[\frac{Z_0 \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_0 \sinh \gamma l} \right]$$

$$\therefore \boxed{Z_S = Z_0}$$

Thus the input impedance of a finite transmission line terminated in its characteristic impedance Z_0, is the characteristic impedance of the line.

→ Input impedance can also be represented as

$$\boxed{Z_S = Z_0 \left[\frac{Z_R + Z_0 \tanh \gamma l}{Z_0 + Z_R \tanh \gamma l} \right]}$$

Infinite Line concepts:—

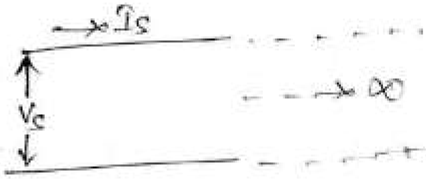
A signal fed into a line of infinite length could not reach the far end in a finite time. Consequently, the condition of the far end (i.e., open and shorted termination) can have no effect at the input end. for this reason transmission line analysis begins with an infinite line inorder to separate input conditions from output conditions.

When an A.C. voltage is applied to the sending end of an infinite line, a finite current will flow due to the capacitance 'C' and the leakage conductance 'G' between the two wires of the line.

The ratio of the voltage applied to the current flowing will give the input impedance of an infinite line. This input impedance is known as characteristic impedance of the line and is denoted by Z_0.

Therefore, $Z_0 = \dfrac{V_S}{I_S}$ ⟶ (1)

Where V_S and I_S are respectively the sending end voltage and current of an infinite line.



→ Current at any point distance x from the sending end is given by

$I = Ce^{-\gamma x} + De^{+\gamma x}$ ⟶ (2)

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The values of 'C' and 'D' can now determined by considering an infinite line.

→ At the sending end of the infinite line, $z=0$ and $I = I_S$.

Applying these conditions to eqn(2), we get

$$I_S = C + D \quad \text{Since } e^0 = 1.$$

However, at the receiving end of the infinite line, $z=\infty$ and $I=0$. Applying these conditions to eqn(2), we get

$$0 = C \times \infty + D \times \infty \qquad \text{Since } e^{\infty} = \infty.$$

(or) $\quad 0 = D \times \infty \qquad$ and $e^{-\infty} = 0$.

Thus either $D=0$ or $\infty=0$. But '∞' cannot be equal to zero. Therefore $D=0$. When $D=0$, $I_S = C$.

Putting the values of 'C' and 'D' in eqn (2), we get

$$I = I_S e^{-\gamma z} \longrightarrow (3)$$

This equation gives current at any point of an infinite line. Similarly the voltage at any point of an infinite line can be deduced to be

$$V = V_S e^{-\gamma z} \longrightarrow (4)$$

Where 'z' is the distance of the point from the sending end of the infinite line.

By definition of propagation constant for a unit length,

$$e^{\gamma} = \frac{I_S}{I_1}$$

where I_1 is the current at a unit distance from sending end. Then, a distance 'z' from sending end, we have

$$e^{\gamma z} = \frac{I_S}{I_R} \longrightarrow (5)$$

Where, I_R is the current at distance 'z'.

But $\gamma = \alpha + j\beta$.

Thus, Eqn (5) becomes,

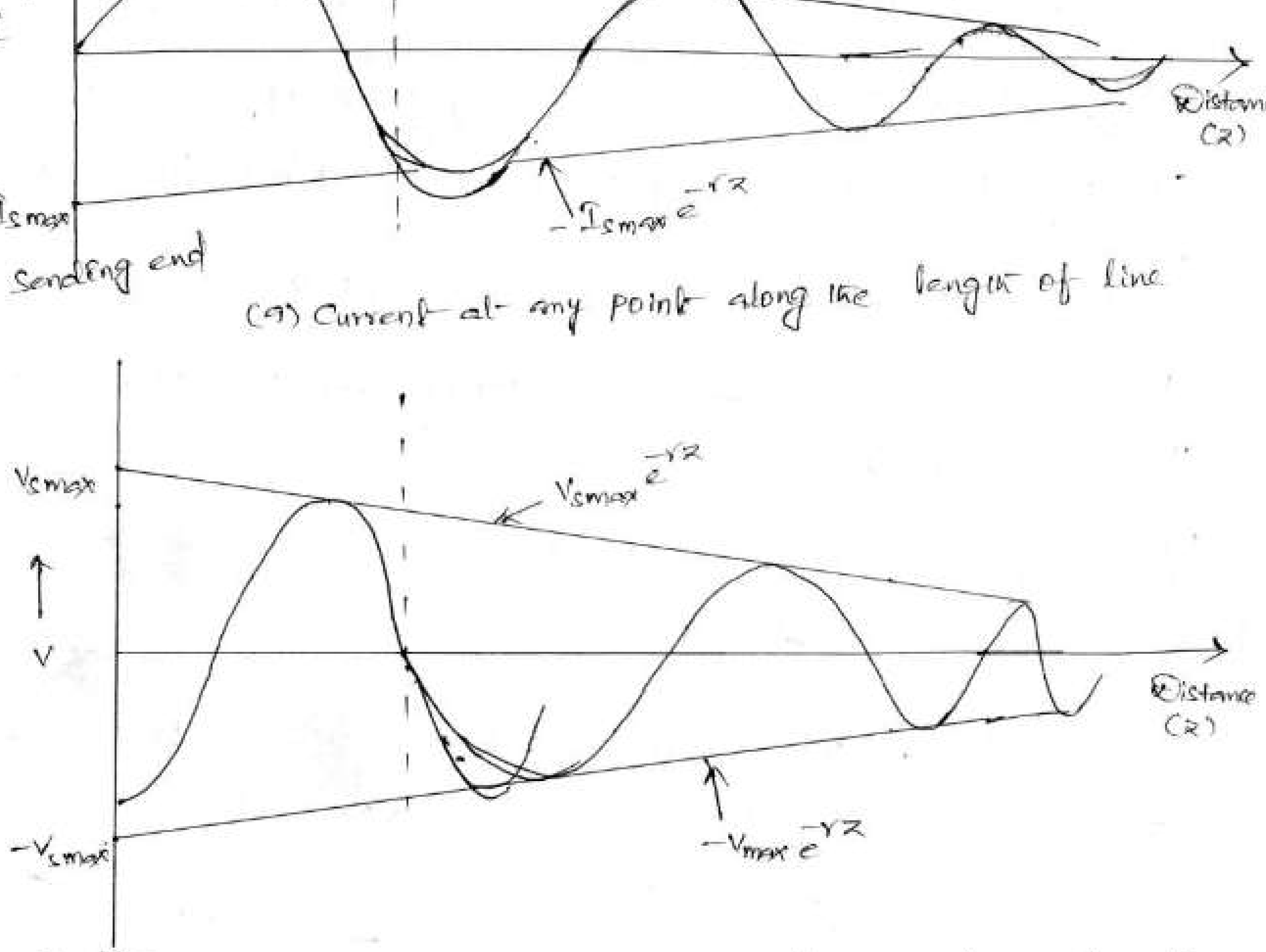
$$\frac{I_S}{I_R} = e^{\gamma z} = e^{\alpha z} \angle \beta z$$

$$\Rightarrow I_R = I_S e^{-\alpha z} \angle -\beta z \longrightarrow (6)$$

Eqn(6) represents the equation of the current at a distance 'z' down the line. Similarly, the voltage V_R, at any point z down the line is given by,

$$V_R = V_S e^{-\alpha z} \angle -\beta z$$

(a) Current at any point along the length of line



(b) voltage at any point along the length of the line

KITL VII (8)

Primary Constants of a Transmission Line :—

For Analysis and design of Transmission Lines, it is necessary to have the knowledge of the electric circuits parameters, associated with the transmission lines.

→ The four line parameters R, L, C and G are termed as primary constants of Transmission Line. They are defined as follows:

Resistance (R) :—

It is defined as Loop Resistance per unit length of line. Thus it is sum of resistances of both the wires for unit line length. It's unit is ohms per km.

→ Depending upon the cross sectional area of the conductors, the transmission line have the resistance associated with them. The resistance is uniformly distributed all along the length of the Transmission Line.

Inductance (L) :—

It is defined as Loop Inductance per unit length of Line. Thus it is sum of inductance of both wires per unit line leng length. It's unit is Henry's per km.

→ When the current passes through the conductor, the magnetic flux is produced around the conductors. The amount of flux depends on the magnitude of the current flowing through the conductor.

Capacitance (C) :—

It is defined as shunt capacitance between the two wires per unit length of the line. It's unit is farad per km.

→ The transmission Line consists of two parallel conductors, separated by a dielectric like air. Such parallel conductors separated by an insulating dielectric produces a capacitance effect.

Conductance (G):— The dielectric in between the conductors is not perfect. Hence a very small amount of current flows through the dielectric called "Displacement Current". This is nothing but a leakage current and this gives rise to a Leakage conductance associated with the Transmission Line.

** It is defined as a shunt conductance between the two wires per unit length of line. It's unit is Mhos per km.

Secondary constants of a Transmission Line:—

→ Two complex constants γ and Z_0 which arose naturally in the process of mathematical simplifications are termed as the secondary constants of the line. γ is called the 'propagation constant' while Z_0 is called the 'characteristic Impedance'.

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{Z}{Y}}$$

and

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{Z \times Y}$$

Although they are referred to as constants but in general all will vary if the frequency is changed.

→ The above equations gives the relationship between primary and secondary constants. These equations helps in calculating secondary constants if the primary constants are known and vice versa.

Characteristic Impedance:— (Z_0)

The characteristic Impedance is the input impedance of an infinite line.

Since we have

$$-\frac{dV}{dz} = (R+j\omega L)\,I \longrightarrow (1)$$

and $V = Ae^{-\gamma z} + Be^{+\gamma z}$

$I = Ce^{-\gamma z} + De^{+\gamma z}$

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if we assume $A = V_f$, $B = V_r$, $C = I_f$, $D = I_r$

$$\therefore \quad V = V_f e^{-\gamma z} + V_r e^{+\gamma z} \longrightarrow (2)$$

$$I = I_f e^{-\gamma z} + I_r e^{+\gamma z} \longrightarrow (3)$$

from eqn (2), $\frac{dV}{dz} = -\gamma V_f e^{-\gamma z} + \gamma V_r e^{+\gamma z} \longrightarrow (4)$

[if there is no reflecting surface, then

$$\frac{dV}{dz} = -\gamma V_f e^{-\gamma z} \Big]$$

from ① & ④,

$$\gamma V_f e^{-\gamma z} - \gamma V_r e^{\gamma z} = (R + j\omega L)\left[I_f e^{-\gamma z} + I_r e^{+\gamma z}\right]$$

at $z = 0$ i.e., at source end,

$$\gamma V_f - \gamma V_r = (R + j\omega L) I \Rightarrow (V_f - V_r) = \frac{R + j\omega L}{\gamma}\left[I_f + I_r\right]$$

$$\Rightarrow \frac{(V_f - V_r)}{I_f + I_r} = \frac{R + j\omega L}{\sqrt{(R + j\omega L)(G + j\omega C)}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

But by definition, the input impedance

$$Z_0 = \frac{V_f - V_r}{I_f + I_r}$$

**

$$\therefore \boxed{Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}}$$

→ characteristic impedance of a uniform transmission line may be defined as the steady-state vector ratio of the voltage to the current at the input of the infinite line. Alternately, it is defined as the impedance looking into an infinite length of the line. It's unit is ohms. It is also known as "Surge Impedance".

Propagation constant :— (γ)

→ The propagation constant per unit length of a uniform line may be defined as the natural logarithm of the steady state vector ratio of the current or voltage at any point, to that at a point unit distance further from the source, when the line is infinitely long.

$$\text{propagation constant } (\gamma) = \sqrt{\text{Series impedance} \times \text{shunt Admittance}}$$

$$\gamma = \sqrt{Z \times Y} = \sqrt{(R+jwL)(G+jwC)}$$

Since $\gamma = \alpha + j\beta$, $\alpha \to$ Attenuation constant
$\beta \to$ phase constant

$$\alpha + j\beta = \sqrt{(R+jwL)(G+jwC)}$$

$$\alpha^2 - \beta^2 + j2\alpha\beta = (R+jwL)(G+jwC) = RG - w^2LC + j(wRC + wLG)$$

By comparing, $\alpha^2 - \beta^2 = RG - w^2LC \longrightarrow (1)$

$2\alpha\beta = w(RC+LG)$

Since $\gamma = \alpha + j\beta$, $|\gamma| = \sqrt{\alpha^2+\beta^2} \longrightarrow (2)$

$\gamma^2 = \alpha^2 + \beta^2$

$\therefore$ from (1), $\alpha^2 + \beta^2 = (R+jwL)(G+jwC) \longrightarrow (3)$

$= \sqrt{(R^2+w^2L^2)(G^2+w^2C^2)}$

from (1) & (3),

$2\alpha^2 = \sqrt{(R^2+w^2L^2)(G^2+w^2C^2)}$

from (1) & (3),

$$2\alpha^2 = (RG - w^2LC) + \sqrt{(R^2+w^2L^2)(G^2+w^2C^2)}$$

$$\therefore \boxed{\alpha = \sqrt{\frac{1}{2}\left[(RG - w^2LC) + \sqrt{(R^2+w^2L^2)(G^2+w^2C^2)}\right]}} \quad **$$

and

$$\boxed{\beta = \sqrt{\frac{1}{2}\left[(w^2LC - RG) + \sqrt{(R^2+w^2L^2)(G^2+w^2C^2)}\right]}} \quad **$$

These equations gives the values of attenuation constant and phase constant in terms of primary constants R, L, G and C.

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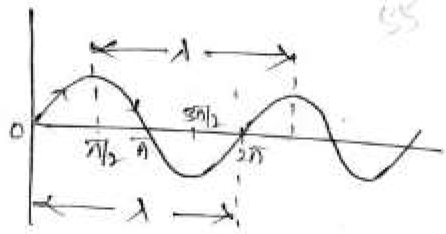
--VII (10)

Wavelength :— It is defined as the distance in which the phase change of 2π radians is effected by a wave travelling along the line as the wave length of a cycle is the distance between two successive positive and negative peaks.



By definition, $\beta\lambda = 2\pi \Rightarrow \boxed{\lambda = \frac{2\pi}{\beta}}$

Phase velocity (or) velocity of propagation (v_p) :—

It is defined as the velocity with which a signal of single frequency propagates along the line at a particular frequency (f). It is measured in km per second. It is the velocity of propagation along the line based on observations of phase change along the line and hence often called as "phase velocity (v_p)". The change of 2π radians in phase angle represents one complete cycle in time t and occurs in a distance of one wavelength (λ), so

$$\lambda = v_p \cdot t = v_p \cdot \frac{1}{f}$$

(or) $$v_p = \lambda \cdot f \qquad \text{where } f \to \text{frequency}$$

$$v_p = \frac{2\pi}{\beta} \cdot f = \frac{\omega}{\beta}$$

(or) $$\boxed{v_p = \frac{\omega}{\beta}} \text{ km/sec.}$$

velocity of propagation of wave can never exceed the velocity of light.

Group velocity :— (v_g)

If the transmission line or transmission medium is such that different frequencies travel with different velocities, then the line or the medium is said to be "dispersive". In that cases, signals are propagated with a velocity known as "Group velocity" (v_g). The group velocity is usually less than the phase velocity. The inter-relation between group velocity and phase velocity can be shown as

$$v_g = \frac{d\omega}{d\beta}$$

(or)

$$v_g = \frac{\omega_2 - \omega_1}{\beta_2 - \beta_1}$$

where 'ω_2' and 'ω_1' are two close angular frequencies being transmitted and β_2 and β_1 the corresponding phase constants. The $d\omega/d\beta$ is measured at the centre frequency.

$$Z_0 = \sqrt{Z/Y} = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$$

$$\boxed{Z_0 \times \gamma = (R+j\omega L)}$$

and $\boxed{\frac{\gamma}{Z_0} = G+j\omega C}$

Lossless Transmission Line:—

→ 'A Transmission Line is said to be lossless if the conductor of the line are perfect i.e., $\sigma_c = \infty$ and the dielectric medium between the lines is lossless i.e., $\sigma_D = 0$. Also a line is said to be lossless if $R=0=G$.

For a Lossless Line,

$$\gamma = \alpha + j\beta = j\beta \qquad \alpha = 0.$$

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$$

as $R=0$, $G=0$, $\gamma = \sqrt{(j\omega L)(j\omega C)} = j\omega\sqrt{LC} = j\beta$

Here $\beta = \omega\sqrt{LC}$, $\alpha = 0$ for Lossless Line.

characteristic Impedance, $Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$

as $R=0=G$, $Z_0 = \sqrt{\frac{L}{C}}$

→ The velocity of propagation in Lossless line, $v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$

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Distortion :—

→ If the output signal at the receiving end is not replica of the input signal at the Transmitting end then there is some distortion present in the Transmission Line.

→ There are different types of Distortions which are occur in Transmission Lines.

→ In General, 'α' is a function of frequency. All frequencies transmitted on a line will then not to be attenuated equally. A complex applied voltage, such as a voice voltage containing many frequencies, will not have all frequencies transmitted with equal attenuation, and the received wave form will not be identical with the input waveform at the sending end. This variation is known as "frequency Distortion"

→ All frequencies applied to a transmission line will not have the same time of transmission, some frequencies being delayed more than others. for an applied voice-voltage wave the received wave form will not be identical with the input wave form at the sending end, since some components will be delayed more than these of other frequencies. This phenomenon is known as 'Delay or Phase Distortion'.

→ frequency Distortion is reduced in the transmission of high quality Radio Broadcast programs over wire lines by use of 'Equalizers'.

→ Delay Distortion is of relatively minor importance to voice and music transmission and it can be very serious circuits intended for Picture transmission and applications of the coaxial cable have been made to overcome the difficulty.

Distortion Less Line :—

A transmission Line is said to be distortionless when the attenuation constant 'α' is frequency independent and phase constant 'β' is Linearly dependent on frequency or when

$$\boxed{\frac{R}{L} = \frac{G}{C}}$$

consider, $\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$

$$\gamma^2 = (R+j\omega L)(G+j\omega C) = (RG - \omega^2 LC) + j\omega(RC + LG) \longrightarrow (1)$$

Since $L = \frac{CR}{G}$ i.e., $LG = CR$

By substituting in eqn (1),

$$\gamma^2 = (RG - \omega^2 LC) + j\omega(2RC)$$

But $RC = LG = \sqrt{RCLG}$

$$\therefore \gamma^2 = RG - \omega^2 LC + 2j\omega\sqrt{RCLG} \Rightarrow \gamma^2 = \left[\sqrt{RG} + j\omega\sqrt{LC}\right]^2$$

$$\therefore \gamma = \sqrt{RG} + j\omega\sqrt{LC}.$$

But $\gamma = \alpha + j\beta$.

$$\therefore \boxed{\begin{array}{l} \alpha = \sqrt{RG} \\ \beta = \omega\sqrt{LC} \end{array}}$$

consider the characteristic impedance (Z_0)

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{R(1+j\omega L/R)}{G(1+j\omega C/G)}}$$

since $L/R = C/G$

$$= \sqrt{\frac{R(1+j\omega L/R)}{G(1+j\omega L/R)}} = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}}$$

$$\boxed{Z_0 = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}}}$$

Since $Z_0 = R_0 + jX_0$

~~Since~~ By comparing, 'j' term is zero and

**
$$\boxed{Z_0 = R_0 = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}}} \rightarrow \text{Distortionless line.}$$

→ The velocity of propagation for distortionless line is

**
$$\boxed{v_p = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}} = \frac{1}{\sqrt{LC}}}$$

1. TTL VII (12)

Overall Transmission Line characteristics:-

Parameter	General Transmission Line	Lossless Line	Distortionless Line
γ	$\sqrt{(R+j\omega L)(G+j\omega C)}$	$j\omega\sqrt{LC}$	$\sqrt{RG}+j\omega\sqrt{LC}$
Z_0	$\sqrt{\frac{(R+j\omega L)}{(G+j\omega C)}}$	$\sqrt{L/C}$	$\sqrt{\frac{L}{C}}$ or $\sqrt{R/G}$
α	$\left[\frac{1}{2}(RG-\omega^2LC)+\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}\right]^{1/2}$	0	$\sqrt{R/G}$
β	$\left[\frac{1}{2}\left[(\omega^2LC-RG)+\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}\right]\right]$	$\omega\sqrt{LC}$	$\omega\sqrt{LC}$
v_p	ω/β	$1/\sqrt{LC}$	$1/\sqrt{L}$

Condition for Minimum Attenuation:—

The attenuation constant 'α' interms of primary constants

$$\alpha=\sqrt{\frac{1}{2}\left[\sqrt{(R^2+\omega^2L^2)(G^2+\omega^2C^2)}+(RG-\omega^2LC)\right]}$$

Hence the attenuation constants depends on the four primary constants along with the frequency under consideration. Thus to find the conditions for minimum attenuation. It is necessary to vary these constants in turn.

Variable 'L':—

Consider 'L' to be variable while R, C and G are the constants for the frequency under consideration.

Hence for Minimum Attenuation, differentiation of α with respect to 'L' must be zero.

$$\therefore \frac{d\alpha}{dL}=0$$

$$\frac{d\alpha}{dL} = \frac{d}{dL}\left\{\frac{1}{2}\left[\left\{(R^2+w^2L^2)(G^2+w^2C^2)\right\}^{1/2} + RG - w^2LC\right]\right\}^{1/2}$$

$$= \frac{1}{2}\left\{\frac{1}{2}\left[\left\{(R^2+w^2L^2)(G^2+w^2C^2)\right\}^{1/2} + RG - w^2LC\right]\right\}^{-1/2} *$$

$$\frac{1}{2}\left\{\left[\frac{1}{2}\left\{(R^2+w^2L^2)(G^2+w^2C^2)\right\}^{1/2-1}\right]\left[2w^2L(G^2+w^2C^2) - w^2C\right]\right\}$$

$$= \frac{1}{2}\;\frac{\frac{1}{2}\left\{\frac{1}{2}\;\frac{2w^2L(G^2+w^2C^2)}{\sqrt{(R^2+w^2L^2)(G^2+w^2C^2)}} - w^2C\right\}}{\sqrt{\frac{1}{2}\left[\sqrt{(R^2+w^2L^2)(G^2+w^2C^2)} + RG - w^2LC\right]}} = 0.$$

The denominator cannot be '0' hence to satisfy this equation.

$$\frac{w^2L(G^2+w^2C^2)}{\sqrt{(R^2+w^2L^2)(G^2+w^2C^2)}} - w^2C = 0$$

$$\Rightarrow \quad \frac{L(G^2+w^2C^2)}{\sqrt{(R^2+w^2L^2)(G^2+w^2C^2)}} = C$$

$$\therefore \quad L\sqrt{\frac{G^2+w^2C^2}{R^2+w^2L^2}} = C \Rightarrow L\sqrt{(G^2+w^2C^2)} = C\sqrt{(R^2+w^2L^2)}$$

$$L^2 = \frac{R^2C^2}{G^2} \Rightarrow L = \frac{CR}{G}$$

When 'L' is variable then the attenuation will be minimum when

$$\boxed{L = \frac{CR}{G} \quad H/Km}$$

In practice 'L' is kept less than this value and hence the attenuation can be reduced by artificially increasing 'L'. This leads to the concept of loading of line.

Similarly for variable 'C'.

$$C = \frac{LG}{R} \quad F/Km$$

In practice. 'C' is normally larger than the value required for minimum attenuation.

AMTL U-VII (13)

R and G for Minimum Attenuation :—

→ When R or G is variable, then by differentiating α and equating it to zero, no minima can be found out mathematically. So theoritically no values of R and G can be obtained for Minimum Attenuation.

But practically it can be seen that when R=0 there are no losses along the line while when G=0 there is no leakage thus when R and G are zero then the attenuation is zero.

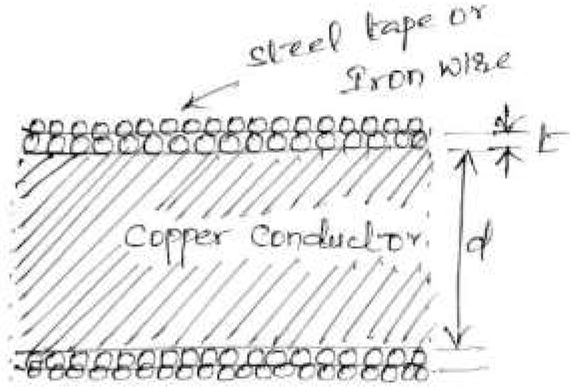
Loading :—

→ To minimize the attenuation and to reduce the distortion we need to increase the value of Inductance i.e., increasing the value of inductance by inserting the Inductance in series with a line is called "Loading" and that lines are called "Loading lines"

→ There are three types of Loading in practice:

(i) continuous Loading
(ii) Patch Loading
(iii) Lumped Loading

Continuous Loading :—



→ This method is used to increase the value of inductance upto 65mH/km but it is expensive.

→ Here a type of iron or some other magnetic material is used and are wound around the conductor to increase the permeability of the surrounding medium and also to increase the value of inductance.

→ The increase in the inductance for a continuously loading line is,

$$\boxed{L \simeq \frac{\mu}{\frac{d}{\pi t}+1}} \text{ mH}$$

where
μ = Permeability of surrounding material
d = Diameter of copper conductor

t = Thickness per Layer of tape or Iron wire

n = No of Layers.

→ The continuously Loaded cable has the advantage over a lumped cable that is the value of 'α' will increase uniformly with the increase in frequency and there will be no cut off frequency.

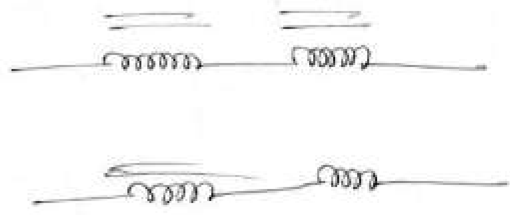
Patch Loading:—

→ This type of Loading is normally known as "Continuously Loaded cable", which separates the section of unloaded cable.

→ In this way the advantage of loading will obtain but the cost is reduced.

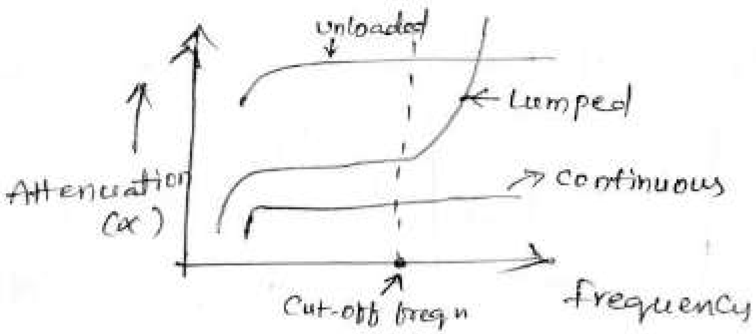
→ In Submarine cable there will be no use of continuous loading over the entire cable since to obtain the reduction in attenuation and a desired result without the continuous loading over whole length of the cable.

Lumped Loading:—



In this type of Loading, the inductors are introduced in series at uniform distances along the line. The Lumped inductors are in the form of coils called "Loading coils".

→ In Lumped Loading the inductance of a line can also be increased by inserting a loading of coils of uniform intervals. This phenomenon is known as "Lumped Loading" and the lumped loaded lines will behave as lowpass filters and this method of Loading is more convenient than the continuous loading.



KITL U-VIII (1)

Open and Short Circuited Lines:—

→ The impedance at any point of the transmission line is the ratio of the voltage to the current, and the impedance also dependent on the type of load at the far end and the distance from it. In other words, in any transmission line the terminating load establishes the current and voltage relations. The various methods by which the voltage and current may be distributed along a transmission line can be simplified by considering three important cases of the terminating load.

(a) When the terminating end is open circuited

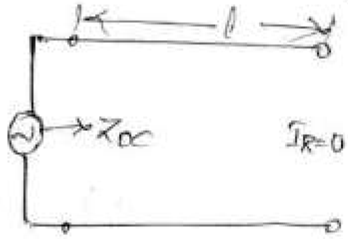
(b) When the terminating end is short circuited.

(c) When the terminating load is equal to the characteristic impedance

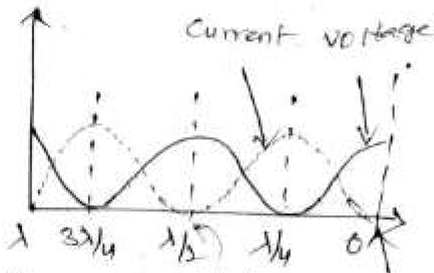
→ By open circuited mean that far end of the transmission line (i.e., at load end) is open while incase of short circuited, far end of the transmission line is shorted by a metallic strip.

→ There are two waves along the line which are travelling in opposite directions between the sending end and receiving ends. They are called as "Incident waves" (travelling from sending to receiving end) and reflected waves (travelling from receiving end to sending end). Because of this there will be interference between the two waves and reinforcement and cancellation will take place. The points where the two waves are in phase, will add and produce voltage maxima, while the points where the they are in opposite phase, will cancel and will produce voltage minima. The points of voltage maxima and voltage minima are also called as "Antinodes" and 'Nodes' respectively. Since in case of open and short circuited lines, the complete reflection occur and thus there is complete cancellation or the voltage minimum is zero.

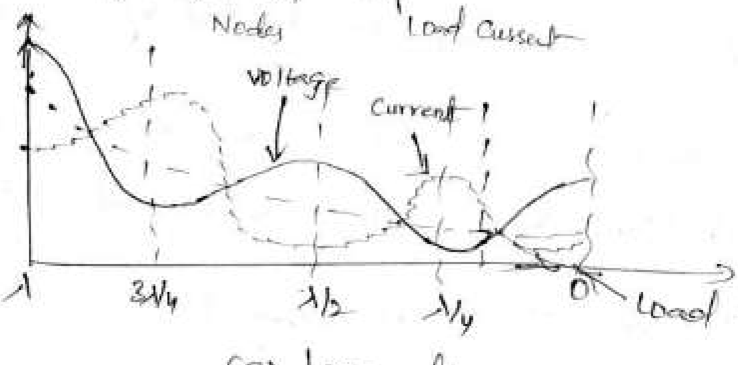
Open Circuited Line:— Since no current can flow in an open circuit, although voltage can exist between the two wires. Therefore, when the line is open circuited, there exists a maximum voltage and minimum current. This implies that impedance of open end is infinite. At a point $\lambda/4$ from the load end the incident wave is 90° earlier while the reflected wave is 90° later than what they are at the load end. Thus there is 180° phase difference between incident and reflected waves, resulting in voltage minimum at the $\lambda/4$ point. The standing wave pattern is repeated every $\lambda/2$ lengths. The current maxima occur at points where the voltages minima and vice versa. At high freq's lossless line, the voltage maxima values are equal but in lossy lines it go on decreasing due to attenuation.



(a) open circuited line



(b) lossless line

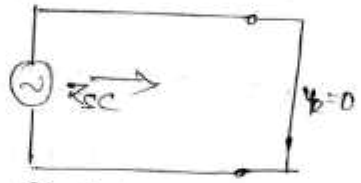


(c) Lossy line

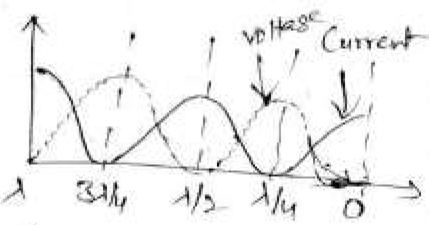
Short Circuited Lines:

When the load end is short circuited, then there is no voltage difference. However, current flows. Thus at short circuited termination, the current is maximum, voltage is zero. So the impedance is zero.

→ for short circuited line, the standing wave have minimum voltage or node at short circuited end and voltage minima are spaced at every $\lambda/2$ apart from the load end. In lossy line, the voltage and current gets attenuated as they travel towards the load end.

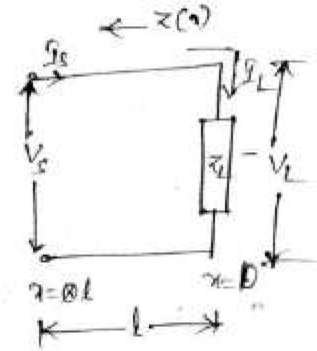


(a) short circuited line



ETTL
U-VIII (2)

Input Impedance of a Transmission Line :—



A transmission line terminated with any load impedance Z_L at $x=0$ as shown in figure.

Equations of voltage and current at a point distance 'x' from sending end are given by

$$V(x) = V_s \cosh\gamma x - I_s Z_0 \sinh\gamma x \longrightarrow (1)$$

and

$$I(x) = I_s \cosh\gamma x - \frac{V_s}{Z_0} \sinh\gamma x \longrightarrow (2)$$

If the line is terminated at a length or distance of $x=l$ with any impedance Z_L then the voltage across the load impedance Z_L is given by

$$V_L = I_L Z_L \longrightarrow (3)$$

The general solution of the line terminated with any impedance Z_L at $x=l$ is given by [eqns (1) & (2)].

$$V_L = V_s \cosh\gamma l - \frac{V_s}{Z_0} \sinh\gamma l \longrightarrow (4)$$

$$I_L = I_s \cosh\gamma l - \frac{V_s}{Z_0} \sinh\gamma l \longrightarrow (5)$$

from eqn's (3), (4) and (5), we have

$$V_s \cosh\gamma l - I_s Z_0 \sinh\gamma l = Z_L\left[I_s \cosh\gamma l - \frac{V_s}{Z_0} \sinh\gamma l\right]$$

$$\Rightarrow V_s \cosh\gamma l + \frac{V_s}{Z_0} \sinh\gamma l \cdot Z_L = Z_L I_s \cosh\gamma l + I_s Z_0 \sinh\gamma l$$

$$\Rightarrow V_s\left[\frac{Z_0 \cosh\gamma l + Z_L \sinh\gamma l}{Z_0}\right] = I_s\left[Z_L \cosh\gamma l + Z_0 \sinh\gamma l\right]$$

$$\Rightarrow \frac{V_s}{I_s} = \frac{Z_0\left[Z_L \cosh\gamma l + Z_0 \sinh\gamma l\right]}{\left[Z_0 \cosh\gamma l + Z_L \sinh\gamma l\right]}$$

since $Z_{in} = V_s/I_s$

$$\therefore Z_{in} = \frac{Z_0\left[Z_L \cosh\gamma l + Z_0 \sinh\gamma l\right]}{\left[Z_0 \cosh\gamma l + Z_L \sinh\gamma l\right]}$$

$$Z_{in} = Z_0 \cdot \frac{\cosh\gamma l \left[Z_L + Z_0 \tanh\gamma l \right]}{\cosh\gamma l \left[Z_0 + Z_L \tanh\gamma l \right]}$$

$$\boxed{Z_{in} = Z_0 \left[\frac{Z_L + Z_0 \tanh\gamma l}{Z_0 + Z_L \tanh\gamma l} \right]} = Z_0 \left[\frac{Z_L \cosh\gamma l + Z_0 \sinh\gamma l}{Z_0 \cosh\gamma l + Z_L \sinh\gamma l} \right]$$

Short Circuit Line:—

If the terminals of the load being shorted i.e., the impedance at the load terminals is zero, then the input impedance of a Transmission line is

$$Z_{in} = Z_0 \left[\frac{Z_L + Z_0 \tanh\gamma l}{Z_0 + Z_L \tanh\gamma l} \right]_{Z_L = 0}$$

$$Z_{in} = Z_{sc} = \frac{\cancel{Z_0} \times Z_0 \tanh\gamma l}{\cancel{Z_0}}$$

$$* \boxed{Z_{sc} = Z_0 \tanh\gamma l}$$

$$\cosh\gamma l = \cos\beta l$$
$$\sinh\gamma l = j\sin\beta l$$
$$\cosh jx = \frac{e^{jx} + e^{-jx}}{2} = \cos x$$

Open Circuit Line:—

If the terminals of the load being opened i.e. the impedance at the load terminals is infinity, then the input impedance of a Transmission line is

$$Z_{in} = Z_0 \left[\frac{Z_L + Z_0 \tanh\gamma l}{Z_0 + Z_L \tanh\gamma l} \right]_{Z_L = \infty}$$

$$Z_{oc} = Z_0 \cdot \frac{\cancel{Z_L}}{\cancel{Z_L}} \left[\frac{1 + Z_0/Z_L \tanh\gamma l}{Z_0/Z_L + \tanh\gamma l} \right]$$

$$= Z_0 \cdot \frac{Z_L}{Z_0} \cdot \frac{1 \tanh\gamma}{\tanh\gamma l} = Z_0 \cdot \frac{1}{\tanh\gamma l}$$

$$** \boxed{Z_{in} = Z_0 \cdot \coth\gamma l = Z_{oc}}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$
$$\sinh jx = \frac{e^{jx} - e^{-jx}}{2}$$
$$= \frac{e^{jx} - e^{-jx}}{2 * j} * j$$
$$= j\sin x$$

$$\sin jx = \frac{e^{x} - e^{-x}}{2}$$
$$j\sinh x = j\left[\frac{e^{x} - e^{-x}}{2}\right]$$
$$= \frac{j^2\left[e^{x} - e^{-x}\right]}{2j}$$
$$= \frac{-e^{x} + e^{-x}}{2j}$$
$$= \frac{-e^{-j(jx)} + e^{+j(jx)}}{2j}$$
$$= \sin jx$$

ETTL U-VIII (3)

$Z_{sc} = Z_0 \tanh \gamma l$

$Z_{oc} = Z_0 \coth \gamma l$ } where $\gamma = \alpha + j\beta$

for a lossless line: $\alpha = 0$, $\gamma = j\beta$.

$\therefore Z_{sc} = Z_0 \tanh j\beta l$

$= Z_0 \frac{\sinh j\beta l}{\cosh j\beta l}$

$Z_{sc} = Z_0 \frac{(j \sin\beta l)}{\cos\beta l} = j Z_0 \tan\beta l$

$(\because \alpha = 0)$

$Z_{oc} = Z_0 \frac{\cosh j\beta l}{\sinh j\beta l}$

$= + Z_0 \frac{\cos\beta l}{j\sin\beta l} = -j Z_0 \cot\beta l$.

$Z_{sc} \cdot Z_{oc} = j Z_0 \tan\beta l \cdot -j Z_0 \cot\beta l$

$= Z_0^2$ $(j^2 = -1)$

$\therefore Z_0 = \sqrt{Z_{sc} \cdot Z_{oc}}$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$\sin jx = j \sinh x$

$\cos jx = \cosh x$

$\sinh jx = j \sin x$

$\cosh jx = \cos x$

$\operatorname{cosech}^2 x + \coth^2 x = 1$

$\cos jx = \frac{e^{j(jx)} + e^{-j(jx)}}{}$

→ Incase of lossy.

$Z_{sc} = Z_0 \tanh \gamma l$, $Z_{oc} = Z_0 \coth \gamma l$

$Z_{sc} \cdot Z_{oc} = Z_0^2 \tanh\gamma l \cdot \coth\gamma l = Z_0^2$

$\Rightarrow Z_0 = \sqrt{Z_{sc} \cdot Z_{oc}}$.

Irrespective of medium of line,

$Z_0 = \sqrt{Z_{sc} \cdot Z_{oc}}$

Z_0 = is a geometrical mean of short circuit and open circuited input impedance.

$$\boxed{Z_{in} = Z_0 \left[\frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l} \right]}$$ (lossy)

$$\boxed{Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \right]}$$ (lossless).

The above expression shows that the input impedance varies periodically with distance 'l' from the load. The quantity βl in the above equation is usually referred to as the 'Electrical length' of the line and can be expressed in degrees or radians.

Reflection coefficient (Γ or K):—

It is defined as the ratio of the reflected voltage to the incident voltage.

→ Reflection coefficient can also be defined as in terms of the ratio of reflected current to incident current. However it is observed that Γ is defined from current ratio is negative with respect to ~~that~~ that defined from voltage ratio. The reason being that the reflected current suffers a 180° phase shift at the receiving end while the reflected voltage doesnot.

→ Reflection coefficient is in general a vector quantity having magnitude and direction both.

→ The ratio of Reflected voltage to incident voltage is called "voltage reflection coefficient", similarly the ratio of reflected current to incident current is "Current Reflection coefficient". The difference between these two is that current reflection coefficient is the negative of voltage reflection coefficient".

$$\boxed{\Gamma = \frac{V_r}{V_i}} \quad \text{or} \quad \Gamma = \frac{-I_r}{I_i}$$

VIII (a)

fundamental equations for voltage and current at any point of transmission line are

$$V = A e^{-\gamma x} + B e^{+\gamma x} \longrightarrow (1)$$

$$I = \frac{1}{Z_0}\left[A e^{-\gamma x} - B e^{+\gamma x}\right] \longrightarrow (2)$$

at $x = 0$, $V = V_L$, $I = I_L$.

$$\therefore V_L = A + B \Rightarrow A + B = V_L \longrightarrow (3)$$

$$I_L = \frac{1}{Z_0}[A - B] \Rightarrow A - B = I_L Z_0 \longrightarrow (4)$$

from (3) & (4),

$$A + \cancel{B} = V_L$$
$$A - \cancel{B} = I_L Z_0$$
$$2A = V_L + I_L Z_0$$
$$A = \frac{V_L + I_L Z_0}{2}$$

fr.

$$\cancel{A} + B = V_L$$
$$\cancel{A} - B = I_L Z_0$$
$$- \quad + \quad -$$
$$2B = V_L - I_L Z_0$$
$$B = \frac{V_L - I_L Z_0}{2}$$

$\therefore$ from ① or ② is

$$V = \left(\frac{V_L + I_L Z_0}{2}\right) e^{-\gamma x} + \left(\frac{V_L - I_L Z_0}{2}\right) e^{\gamma x}$$

$$I = \frac{1}{Z_0}\left[\left(\frac{V_L + I_L Z_0}{2}\right) e^{-\gamma x} - \left(\frac{V_L - I_L Z_0}{2}\right) e^{\gamma x}\right]$$

Since we have $\Gamma = \frac{\text{Reflected wave}}{\text{Incident wave}}$

$$\Gamma = \frac{(V_L - I_L Z_0)/2}{(V_L + I_L Z_0)/2} \frac{e^{\gamma x}}{e^{-\gamma x}} \qquad \left(\text{at } x = 0,\ e^{\gamma x} = e^{-\gamma x} = 0\right)$$

$$\Gamma = \frac{V_L - I_L Z_0}{V_L + I_L Z_0} = \frac{\cancel{I_L}\left[V_L/I_L - Z_0\right]}{\cancel{I_L}\left[V_L/I_L + Z_0\right]} = \frac{V_L/I_L - Z_0}{V_L/I_L + Z_0}$$

$$\boxed{\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}} \quad \text{where } Z_L = \frac{V_L}{I_L}$$

$\Gamma \rightarrow$ It is a measure of mismatch between the expected line impedance Z_0 and the actual impedance at the point $z(x)$.

Four cases of complete mismatch on a line:—

case(i):— Line is lossless: $[\text{i.e., } Z_0 = R_0]$ and load is inductive:

$Z_L = jR_0$ = Inductive $\quad \therefore \Gamma = \dfrac{jR_0 - R_0}{jR_0 + R_0} = \dfrac{j-1}{j+1} = \dfrac{\sqrt{2}\angle 135^\circ}{\sqrt{2}\angle 45^\circ}$

forward voltage → sine wave $\quad \Gamma = 1\angle 90^\circ$

Reflected voltage → cos wave $\quad \therefore \boxed{\Gamma = j}$

case(ii):— Line is Lossless, Load capacitive:

$Z_L = -jR_0$ = capacitive

$\Gamma = \dfrac{-jR_0 - R_0}{-jR_0 + R_0} = \dfrac{-j-1}{-j+1} = \dfrac{j+1}{j-1} = 1\angle -90^\circ$

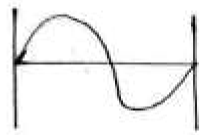
$\boxed{\Gamma = -j}$

forward → sin
Reflected → −cos

case(iii):— Line is lossless, load open circuited:

$[\text{i.e., } Z_0 = R_0, \; Z_L = \infty = O/C]$

$\Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0} = \dfrac{1 - Z_0/Z_L}{1 + Z_0/Z_L} = 1$



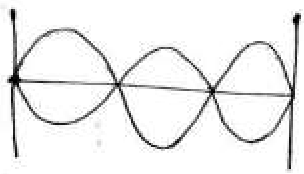
$\therefore \Gamma = 1$

forward → sin
Reflected → sin

case(iv):— Line is lossless, Load is short circuited:

$[\text{i.e., } Z_0 = R_0, \; Z_L = 0 = S/C]$

$\therefore \Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0} = -1$



$\therefore \Gamma = -1$

$= 1\angle 180^\circ$

forward → Sin
Reflected → −Sin

MTTL U-VIII (5)

→ Three specific load conditions that frequently are (i) matched load i.e., $Z_R = Z_0$ (ii) short circuit load, (iii) open circuit load. For these,

(a) Matched load, $Z_L = Z_0$. $\Gamma = 0$

(b) Short circuit at load end, $Z_L = 0$, $\Gamma = -1$

(c) Open circuit at load end i.e., $Z_L = \infty$.

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{\cancel{Z_L}\left[1 - Z_0/Z_L\right]}{\cancel{Z_L}\left[1 + Z_0/Z_L\right]} = 1 \quad \text{as } Z_L = \infty.$$

→ The range of Γ is $-1 \leq \Gamma \leq 1$

$$\boxed{0 \leq |\Gamma| \leq 1}$$

Voltage Standing Wave Ratio (VSWR) :—

→ The plot of the magnitudes of two waves travelling in opposite directions and resulting in addition and cancellation defines a "standing wave". The standing wave of current and voltage with definite maxima and minima along the line as shown. Even when the reflection takes place in line transmission at some points, the incident and reflected signals are in phase and both the components together and at some other points, the two components may oppose each other. The points where the resultant signal (voltage or current) is maximum are known as 'voltage or current maxima". On the other hand the points where the resultant signal (voltage or current) is minimum are called 'voltage or current minimum".

Thus it is evident that,

$$|V_{max}| = |V_i| + |V_r|$$

$$|V_{min}| = |V_i| - |V_r|$$

$$|I_{max}| = |I_i| + |I_r|$$

$$|I_{min}| = |I_i| - |I_r|$$

The magnitude of standing wave provides an idea of the amount of reflection.

→ The ratio of the maximum and minimum magnitudes of current or voltage on a line having standing waves is called the standing wave ratio, generally denoted by the letter 'S'.

$$\therefore \ VSWR = \frac{|V_{max}|}{|V_{min}|} \quad , \quad CSWR =$$

$$\because \ \cancel{V_{max} = \frac{1+|\Gamma|}{1}} \quad |V_{max}| = |V_i| + |V_r|$$

$$|V_{min}| = |V_i| - |V_r|$$

$$\therefore \ VSWR = \frac{|V_i| + |V_r|}{|V_i| - |V_r|} = \frac{\cancel{|V_i|}\left[1 + |V_r/V_i|\right]}{\cancel{|V_i|}\left[1 - |V_r/V_i|\right]} = \frac{1+|\Gamma|}{1-|\Gamma|}$$

$$\therefore \ \boxed{VSWR = \frac{1+|\Gamma|}{1-|\Gamma|}}$$

The points at which voltage is maximum, current is minimum and vice versa. So $\Gamma = \frac{|I_{max}|}{|I_{min}|}$

Similarly, $CSWR = \frac{|I_{max}|}{|I_{min}|}$

$$\because \ I_{max} = |I_i| + |I_r|$$

$$I_{min} = |I_i| - |I_r|$$

$$\therefore \ CSWR = \frac{|I_i| + |I_r|}{|I_i| + |I_r|} = \frac{\cancel{|I_i|}\left[1 + |I_r/I_i|\right]}{\cancel{|I_i|}\left[1 + |I_r/I_i|\right]}$$

$$CSWR = \frac{1 + |I_r/I_i|}{1 + |I_r/I_i|}$$

$$\therefore \ \boxed{CSWR = \frac{1+|\Gamma|}{1+|\Gamma|}} \qquad \text{Where } \Gamma = \frac{I_r}{I_i}$$

→ It is significant that VSWR is always greater than 1, and when it is equal to 1, the line is correctly terminated and there is no reflection.

→ The range of 'S' is from 1 to ∞.

KTTL
U-VIII (6)

Relation between 'S' and 'Γ' :—

$$(VSWR)\ S = \frac{1+|\Gamma|}{1-|\Gamma|}$$

$$S - S|\Gamma| = 1+|\Gamma| \Rightarrow |\Gamma| + S|\Gamma| = 1-S$$

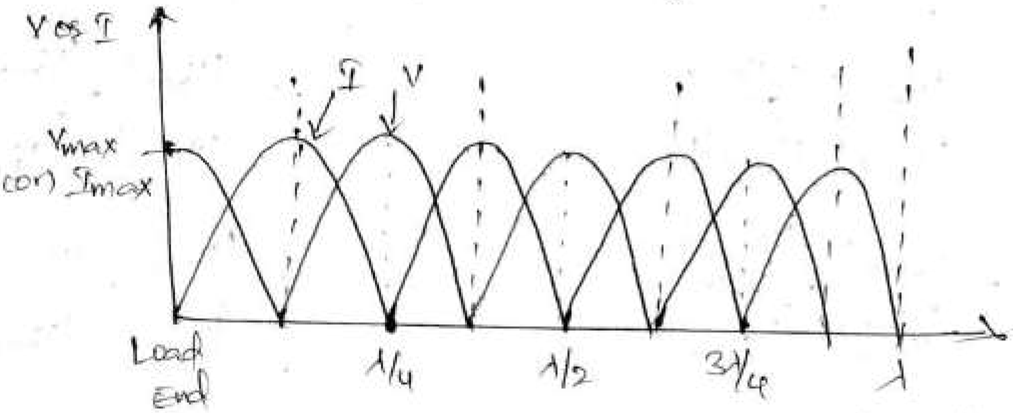
$$|\Gamma|[1+S] = 1-S$$

$$\Rightarrow \boxed{|\Gamma| = \frac{1-S}{1+S}}$$

→ Total reflection can takes place in two cases i.e., zero load or infinite load. These two cases are

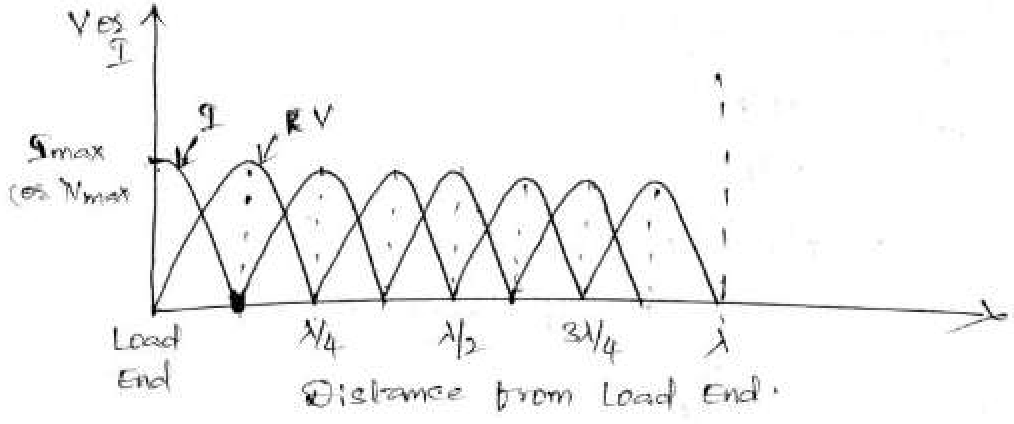
(i) open circuit Load :—

Incase of open circuit load, the load current is zero. It means that incident current I_i and the reflected current 'I_r' at the load are equal in magnitude but opposite in phase so that they cancel each other and net current becomes zero. Thus we have a current minima at the load point. This point corresponds to voltage maxima. At a distance of $\lambda/4$ from the load end, the conditions are opposite and we have a current maxima / voltage minima. This pattern repeats all along the line.



(ii) short circuit Load :—

When load is short circuit, load current is maximum but load voltage is zero. Thus we have a point of voltage minima / current maxima at the termination. Again the conditions change after $\lambda/4$ and point of voltage maxima / current minima is reached. This pattern is repetitive along the line.



*→ The periodicity of standing wave is $\lambda/2$ (i.e., every cycle is repeated for every $\lambda/2$ distance from the load end)

*→ The spacing between two successive maximas or minimas is "$\lambda/2$" and the spacing between successive maxima and minima is "$\lambda/4$".

*→ Range of SWR is $[1, \infty]$

SWR = 1 at perfectly matched.

SWR = ∞ at perfectly mismatched.

---

*→ Z_{max} and Z_{min} :

$$Z_{max} = \frac{|V_{max}|}{|I_{min}|} = \frac{|V_{max}|}{|I_{min}|} \times \frac{\left|\frac{V_{min}}{I_{max}}\right|}{\left|\frac{V_{min}}{I_{max}}\right|} = \frac{|V_{max}|}{\left|\frac{V_{min}}{I_{max}}\right|} \cdot \frac{\left|\frac{I_{max}}{V_{min}}\right|}{|I_{min}|}$$

$$Z_{min} = \frac{|V_{min}|}{|I_{max}|} = \frac{|V_{min}|}{|I_{max}|} \times \frac{|V_{max}|}{|V_{max}|} = \frac{|V_{max}|}{|I_{max}|} \cdot \frac{|V_{min}|}{|V_{max}|}$$

***
$$\boxed{Z_{max} = Z_0 \,[VSWR]} \qquad \boxed{\frac{V}{I} = Z_0}$$

$$Z_{min} = Z_0 \cdot \frac{1}{(VSWR)}$$

***
$$\boxed{Z_{min} = \frac{Z_0}{VSWR}}$$

$$Z_{max} = \frac{V_{max}}{I_{min}} = \frac{|V_i| + |V_R|}{\frac{|V_i| - |V_R|}{Z_0}}$$

$$= Z_0 \cdot \frac{|V_i|\,[1 + |V_R/V_i|]}{|V_i|\,[1 - |V_R/V_i|]}$$

$$Z_{max} = Z_0 \cdot \left(\frac{1+|\Gamma|}{1-|\Gamma|}\right) \Rightarrow \boxed{Z_{max} = Z_0 \cdot VSWR}$$

ETTL
U-VIII (7)

UHF (Ultra High frequency) Lines:—

Ultra High frequency lines normally abbreviated as UHF lines covers frequency range from 300MHz to 3GHz. Therefore, UHF lines are those transmission lines which are operate from 300MHz to 3GHz or whose wavelengths are from 100cm to 10cm.

→ Various lines which are operating at the frequency range of 300MHz to 3GHz. Generally they are

- → $\lambda/8$ (Eighth-wave line) Lines
- → $\lambda/4$ (Quarter-wave) Lines
- → $\lambda/2$ (Half-wave) Lines.

$\lambda/8$ (Eighth-wave Line):—

The input impedance of a Eighth wave line is

$$Z_{in} = Z_0\left[\frac{Z_L\cos\beta l + jZ_0\sin\beta l}{Z_0\cos\beta l + jZ_L\sin\beta l}\right]$$

$$l=\lambda/8,\quad \beta l = \frac{2\pi}{\lambda}\times\lambda/8 = \frac{\pi}{4}$$

$$\therefore Z_{in} = Z_0\left[\frac{Z_L\cos\pi/4 + jZ_0\sin\pi/4}{Z_0\cos\pi/4 + jZ_L\sin\pi/4}\right] = Z_0\left[\frac{Z_L\cdot 1/\sqrt{2} + jZ_0\cdot 1/\sqrt{2}}{Z_0\cdot 1/\sqrt{2} + jZ_L\cdot 1/\sqrt{2}}\right]$$

**

$$\boxed{Z_{in\,(\lambda/8)} = Z_0\left[\frac{Z_L + jZ_0}{Z_0 + jZ_L}\right]}$$

If Eighth wave (or $\lambda/8$) transmission line is terminated with the load impedance (Z_L) then $Z_L = Z_0$.

$$\therefore Z_{in}(l=\lambda/8) = Z_0\left[\frac{Z_0 + jZ_0}{Z_0 + jZ_0}\right] = Z_0$$

$$\therefore \boxed{Z_{in}(l=\lambda/8) = Z_0}$$

Thus an Eighth wave line may be used to transform any resistance to an impedance with a magnitude equal to Z_0 of the line.

$\frac{\lambda}{4}$ line (or Quarter wave line):—

The expression for the input impedance of a $\lambda/4$ transmission line is

$$Z_{in} = Z_0 \left[\frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l} \right]$$

$l = \lambda/4$, $\beta l = \frac{2\pi}{\lambda} * \frac{\lambda}{4} = \pi/2$.

$$Z_{in} = Z_0 \left[\frac{Z_L \cos\pi/2 + jZ_0 \sin\pi/2}{Z_0 \cos\pi/2 + jZ_L \sin\pi/2} \right]$$

$$= Z_0 \left[\frac{\cancel{j} Z_0 \sin\pi/2}{\cancel{j} Z_L \sin\pi/2} \right] = \frac{Z_0^2}{Z_L}$$

$$\therefore \boxed{Z_{in} = \frac{Z_0^2}{Z_L}}$$

That is, the input impedance of the line is equal to the square of Z_0 of the line divided by the load impedance.

→ A quarter-wave section of line is used as a transformer to match a load of Z_L ohms to a source of Z_{in} ohms.

→ An important application of the Quarter-wave matching section is to couple a transmission line to a resistive load such as an antenna. The quarter-wave matching section must be designed to have a characteristic impedance Z_0' so chosen that the antenna resistance Z_A is transformed to a value equal to the characteristic impedance Z_0 of the transmission line. Then the line is terminated in its Z_0 and is operated under conditions of no reflection.

→ The characteristic impedance Z_0' of the matching section should be

$$\boxed{Z_0 = \sqrt{Z_{in} \cdot Z_L}}$$

→ For a loss less line, $\lambda/4$ line acts as an excellent "Impedance Inverter".

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λ/2 (Half-wave) Line:—

When a length of line having $l=\lambda/2$, the input impedance of λ/2 line is

$$Z_{in} = Z_0\left[\frac{Z_L\cos\beta l + jZ_0\sin\beta l}{Z_0\cos\beta l + jZ_L\sin\beta l}\right]$$

When $l=\lambda/2$, $\beta l = \frac{2\pi}{\lambda}\times\lambda/2 = \pi$

$$\therefore Z_{in(\lambda/2)} = Z_0\left[\frac{Z_L\cos\pi + jZ_0\sin\pi}{Z_0\cos\pi + jZ_L\sin\pi}\right] = Z_0\left[\frac{-Z_L+0}{-Z_0+0}\right]$$

$$Z_{in(\lambda/2)} = \frac{Z_0 Z_L}{Z_0} = Z_L$$

$$\therefore \boxed{Z_{in} = Z_L} \;**$$

A half wavelength of line may be considered as a one-to-one transformer. It has its greatest utility in connecting a load to a source in a case where the load and source cannot be made adjacent.

→ For a lossless line, a λ/2 line is called an "Impedance Reflector".

Smith chart:—

→ Smith chart is a circular chart, in this smith chart R and X are represented in circular form.

Normalized scaling allows the smith chart to be used for problems involving any characteristic system impedance.

Since we have, $\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1}$

Thus $\Gamma = \frac{z_L - 1}{z_L + 1}$ where z_L → normalised terminating impedance

$$z_L = \frac{Z_L}{Z_0}$$

Hence $z_L = \frac{1+\Gamma}{1-\Gamma}$ ⟶ (1)

Since Z_L and Γ both are complex quantities, we have

$$R+jX = \frac{1+\Gamma_r+j\Gamma_x}{1-(\Gamma_r+j\Gamma_x)}$$

Rationalising the right-hand side, we get

$$R+jX = \frac{1+\Gamma_r+j\Gamma_x}{(1-(\Gamma_r)+j\Gamma_x)} \times \frac{(1+(\Gamma_r)+j\Gamma_x)}{1+(\Gamma_r)+j\Gamma_x)}$$

$$= \frac{1+\cancel{\Gamma_r}+\cancel{j\Gamma_x}+\cancel{\Gamma_r}+\Gamma_r^2+j\cancel{\Gamma_r}\Gamma_x+j\cancel{\Gamma_x}+j\Gamma_r\cancel{\Gamma_x}+\Gamma_x^2}{1-(\Gamma_r+j\Gamma_x)^2 \quad (1-\Gamma_r)^2+\Gamma_x^2}$$

$$R+jX = \frac{1+2\Gamma_r-\Gamma_x^2+\Gamma_r^2+j(2\Gamma_x+2\Gamma_r\Gamma_x)+\ldots)}{(1-\Gamma_r)^2+\Gamma_x^2}$$

By equating Real and Imaginary parts,

$$R = \frac{1-\Gamma_r^2-\Gamma_x^2+}{(1-\Gamma_r)^2+\Gamma_x^2} \rightarrow (i) \quad , \quad X = \frac{2\Gamma_x}{(1-\Gamma_r)^2+\Gamma_x^2} \rightarrow (ii) \longrightarrow (2)$$

Eqn (2) yields two sets of circles when solved separately. Eqn (2)(i) result in a family of circles called 'R-circles' while Eqn (2)(ii) result in a family of circles called 'X-circles'.

The constant R-circles:—

first taking eqn (1) and cross-multiplying, we get

$$R(1-\Gamma_r)^2+R\Gamma_x^2 = 1-\Gamma_r^2-\Gamma_x^2$$

$$\Rightarrow R+R\Gamma_r^2-2R\Gamma_r+\Gamma_x^2R = 1-\Gamma_r^2-\Gamma_x^2$$

$$\Rightarrow \Gamma_r^2+R\Gamma_r^2+1+R-2R\Gamma_r+\Gamma_x^2+R\Gamma_x^2=0.$$

$$\Gamma_r^2(1+R)+\Gamma_x^2(1+R)-2R\Gamma_r = 1-R$$

$$\Rightarrow \Gamma_r^2+\Gamma_x^2-\frac{2R\Gamma_r}{1+R} = \frac{1-R}{1+R}$$

Adding $\frac{R^2}{(1+R)^2}$ to both the sides to make it a perfect square, we have

$$\Gamma_x^2+\Gamma_r^2-\frac{2R\Gamma_r}{1+R}+\frac{R^2}{(1+R)^2} = \frac{1-R}{1+R}+\frac{R^2}{(1+R)^2}$$

$$\left(\Gamma_r-\frac{R}{R+1}\right)^2+\Gamma_x^2 = \frac{1-R^2+R^2}{(1+R)^2} = \frac{1}{(1+R)^2}$$

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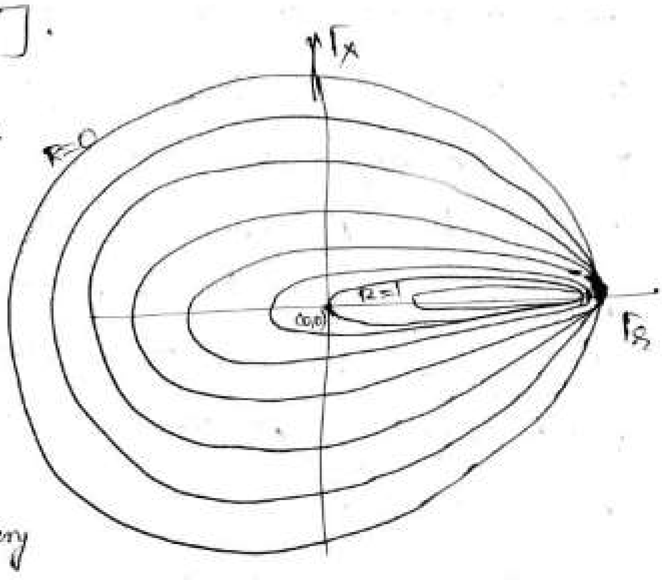
$$\boxed{\left(\Gamma_R - \frac{R}{R+1}\right)^2 + \Gamma_x^2 = \left(\frac{1}{1+R}\right)^2} \longrightarrow (3)$$

Eqn(3) represents family of circles on the reflection coefficient plane. These circles are called 'constant R circles', having radius $\frac{1}{(1+R)}$ and centre $\left[\frac{R}{(1+R)}, 0\right]$.

These circles have their centres on the positive Γ_R axis and are contained in the region 0 to 1 as shown in figure. R=0 corresponds to a circle with centre (0,0) on the Γ-plane. This circle forms the periphery of the smith chart. All constant R-circles touch the point (1,0) including that of R=∞ which is the same as the point itself.



family of constant circles.

The constant X-circles :—

Taking now eqn 2(b) and cross multiplying we get

$$X = \frac{2\Gamma_x}{(1-\Gamma_R)^2 + \Gamma_x^2}$$

$$\Rightarrow X + X\Gamma_R^2 - 2X\Gamma_R + X\Gamma_x^2 = 2\Gamma_x$$

$$\Rightarrow \left[1+\Gamma_R^2 - 2\Gamma_R + \Gamma_x^2\right] = \frac{2\Gamma_x}{X}$$

$$\Rightarrow (1-\Gamma_R)^2 + \Gamma_x^2 - \frac{2\Gamma_x}{X} = 0$$

Adding $(1/X)^2$ on both the sides of the above equation,

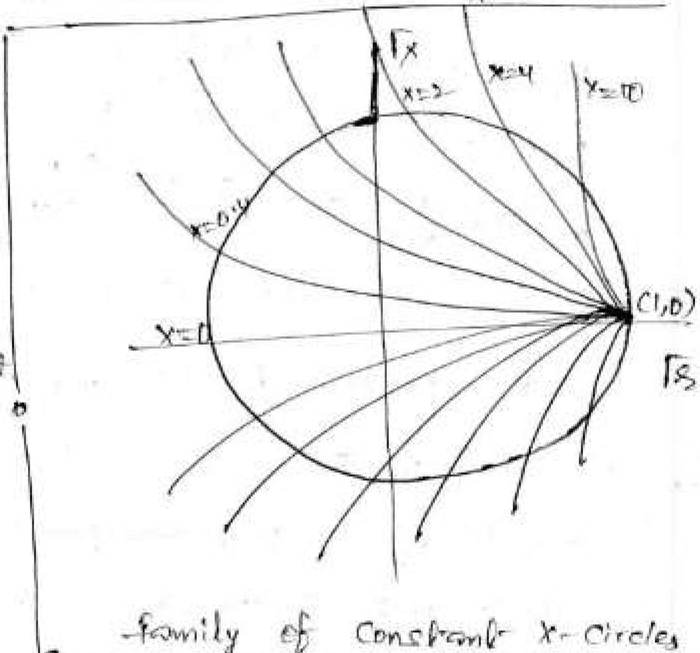
$$(1-\Gamma_R)^2 + \Gamma_x^2 - \frac{2\Gamma_x}{X} + 1/X^2 = (1/X)^2$$

$$\Rightarrow (1-\Gamma_R)^2 + (\Gamma_x - 1/X)^2 = (1/X)^2$$

$$\Rightarrow \boxed{(\Gamma_R - 1)^2 + (\Gamma_x - 1/X)^2 = (1/X)^2} \longrightarrow (4)$$

Eqn(4) represents another family of circles called 'Constant X-Circles' with centre (1, 1/X) and radius 1/X on the Γ-plane as shown in figure.

X being the reactance can be positive or negative. whenever 'X' is positive, the circle lies above the horizontal line i.e. $\Gamma_x = 0$ (or the real axis). On the other hand when 'X' is negative, the circle lies below the real axis $\Gamma_x = 0$. When $x = 0$ the circle degenerates into a straight line $\Gamma_x = 0$, because straight line is a circle whose radius is infinity and for $x=0$, the radius $1/x$ will be infinity. All circles touch the point (1, 0).



family of Constant X-Circles

The Complete Smith chart is obtained by the Superposition of the two sets of X-circles and R-circles.

properties of Smith Chart:—

(i) Normalising Impedance:—

The process of dividing impedance Z_L by Z_0 is called 'Normalising an impedance'. Z_0 is the characteristic impedance of the line, which for a lossless line is a pure resistance. If Z_0 is not given in a problem to be solved by smith chart, a suitable value is assumed, for the purpose of normalising given impedance.

The process of normalisation is reversed if a certain impedance is taken from smith chart i.e. this impedance will be multiplied by Z_0.

(ii) plotting of an Impedance:—

Any complex impedance can be shown by a single point on the Smith chart. This point will be the point of intersection of R/Z_0 circle and jX/Z_0 circle.

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(iii) Determination of SWR:—

After having located the point 'P' corresponding to the given impedance, the S-circle can be drawn with 'O' as centre and OP as radius. In order to find the VSWR from the chart, we follow the S-circle around to its right hand intersection with the horizontal axis i.e. the point 'M'. The normalised resistance at the point 'M' is numerically equal to the voltage standing wave ratio 'S'.

(iv) Determination of K in Magnitude and Direction:—

OP is produced till it cuts the angle of reflection coefficient circle at 'N', the reading of this will give the angle of reflection co-efficient. The line ON, in fact is the Γ-scale, giving the magnitude of 'Γ'.

(v) Location of Voltage Maximum and Minimum:—

The intersection of the S-circles with horizontal axis AB, that is on the left of the chart centre represents voltage minima, intersection with the horizontal axis on the right of the centre corresponds to voltage maxima. Thus 'M' and R respectively gives the position of voltage maxima and voltage minima.

(vi) Open and Short circuited Line:—

At the right hand end of the horizontal axis, i.e., the value of 'R' and 'X' both are infinity. Thus, impedance at 'B' is infinity. Since at the open circuit termination $Z_L = \infty$. The position B represents the open circuited termination of a line.

Similarly, at the left hand end of the horizontal axis i.e., 'A', the value of 'R' and 'X' both are zero. Thus impedance at 'A' is zero. Since at the short circuit termination $Z_L = 0$. The position 'A' represents the short circuited termination of the line.

(vii) Matched Load:— At every point on the circle, $\frac{R}{Z_0} = 1 \Rightarrow R = Z_0$. which shows that resistive component of terminating impedance is equal to the characteristic impedance of the line.

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When R=1 circle passes through the centre (1,0) of the smith chart the reactive component is zero. That is,

$$z_r = \frac{Z_L}{Z_0} = 1+j0$$

(or) $Z_L = Z_0$

Therefore, the centre of the chart is known as "Matched Load point" and condition of no reflection is completely satisfied since the terminating impedance is equal to the characteristic impedance of the line.

Application of Smith chart:

(i) Smith chart as an Admittance Diagram:

The Smith chart can be used as an admittance diagram.

The normalised input admittance y is given by,

$$Y = g - jb \longrightarrow (1)$$

where g is the normalised conductance (reciprocal of resistance) and b is the normalised susceptance (reciprocal of reactance).

Since we have, $Y = \frac{1}{z}$

$$g - jb = \frac{1}{r+jx} = \frac{1}{r+jx}\cdot\frac{r-jx}{r-jx} = \frac{r-jx}{r^2+x^2}$$

Thus, $$g - jb = \frac{r}{r^2+x^2} - j\frac{x}{r^2+x^2} \longrightarrow (2)$$

This is also reciprocal to Γ and hence

$$g - jb = \frac{1-\Gamma}{1+\Gamma} \longrightarrow (3)$$

Since we have $$\Gamma = \frac{z_r - 1}{z_r + 1} \longrightarrow (4)$$

By comparing the above two eqns, Eqn(3) is obtainable from eqn(4) by replacement of r by g, x by −b and k by −k.

(ii) converting impedance into Admittance:—

→ In a lossless line of $\lambda/4$ length, the product of the input impedance and the load impedance is equal to the square of the characteristic impedance Z_0. Thus,

$$Z_{IN} * Z_L = Z_0^2$$

$$\frac{Z_{IN}}{Z_0} = \frac{Z_0}{Z_L}$$

$$z_{in} = \frac{1}{z_L}$$

where z_{in} and z_L are the normalised input and load impedance.

$$\boxed{z_{in} \cdot z_L = 1.}$$

Therefore, inorder to find the admittance of an impedance, the impedance point has to be rotated to a distance of $\frac{\lambda}{4}$ (or) 0.25λ since the 0.25λ rotation in a Smith chart corresponds to half the cycle, the admittance simply be a point diametrically opposite to the impedance point.

(iii) Determination of an input impedance:—

Consider a load impedance expressed by a point P on the Smith chart, with O as centre and OP as radius S circle can be drawn. produce OP to cut wavelength scale at Q. Thus standing at Q is equivalent to standing at the terminating impedance i.e. load as shown. Inorder to find the input impedance of length l, we have to go towards the generator i.e., clockwise on the Smith chart. This way locate the point T on the wavelength scale at a distance $\frac{l}{\lambda}$ from S. Join OT which will give input R on the S circle. The point will give the normalised input impedance which, when multiplied by Z_0 will give the input impedance.

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(iv) Determination of the Impedance:—

Smith chart can also be used to determine load impedance if SWR and the distance of first voltage minima from the load.

Since the point of Voltage minima always lies in the left-hand side of the horizontal axis at a distance $\frac{1}{s}$ from the centre O of the chart; locate V_{min} on the chart. Move the given distance of V_{min} from A and locate the point Q on the wavelength chart. Join QO to cut the circle at P. The coordinates of P will give the normalised load impedance which when multiplied by Z_0 will give the desired load impedance.

(v) Determination of Input Impedance and Admittance of short circuited line:—

Since the input impedance is purely reactive it has to be read on the circle corresponding to the real part equal to zero. which is periphery of the smith chart. Each point on this circle is associated with a particular value of βl marked in wavelengths. Hence, the value read on this circle gives the normalised input impedance of a given short circuited line, the starting point being x=0.

(vi) Determination of Input Impedance and Admittance of an open circuited line:—

→ for the determination of the input impedance and admittance of an open circuited, the same procedure as above is adopted except that the angle of marking on the chart is shifted by $\frac{\pi}{2}$ radians. Since there exists a phase difference of $\frac{\pi}{2}$ radians between impedance (or admittance) of the short circuited line and that of the open circuited line.

Note:—

→ All the X-Circles are concurrent and and have a common tangent Γ_{real} axis.

→ If X is from 0 to 1 (0<X<1). The circles lie only in the I and II Quadrants only but when X is greater than 1 they lie in I Quadrant only.

→ If −1<X<0, the circle lie in 3rd & 4th Quadrants only and when X<−1, the circles lie in the fourth Quadrant only.

→ R and X always constitutes a pair of orthogonal planes circles.

→ As the centres of R circles lie on the common tangent of X circles and vice versa.

→ By moving through R=0 circles, X values increases and moving through X=0 circle R value increases.

→ A complete revolution (360°) around the smith chart represents a distance of $\lambda/2$ on the line.

Since a $\lambda/2$ distance on the line corresponds to a movement of 360° on the chart, λ distance on the line corresponds to a 720° movement on the chart.

$$\lambda \rightarrow 720°$$

→

Stub Matching:—

→ A section of transmission line can be used as matched section by inserting them between load and the source. Besides it is also possible to connect sections of open or short circuited line known as "stub or tuning stub" in shunt with the main line at a certain point (or points) to effect the matching. The matching with the help of tuning stub or stub is called "stub matching" and is having the following advantage.

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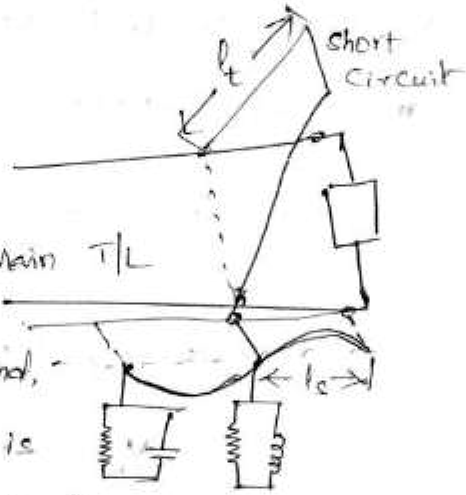
(i) Length (l) and characteristic impedance (Z_0) remains unchanged.

(ii) It is possible to add adjustable susceptance in shunt with the line.

A stub matching is of two types, the single stub matching and double stub matching.

Single stub Matching:—



The principal element of this transformer is a short circuited section of line whose open end connected to the main line at a particular distance from the load end, where the input conductance at that point is equal to the characteristic conductance of the line, and the stub length is adjusted to provide a susceptance which is equal in value but opposite in sign, to the input susceptance of the main line at that point, so that the total susceptance at the point of attachment is zero. The combination of stub and line will thus present a conductance which is equal to the characteristic conductance of the line.

Let us consider a transmission line having a characteristic admittance Y_0 terminated in a pure conductance Y_L. When Y_L is different from Y_0 we know that standing waves are set-up. Related to the input impedance, as we traverse the line from the load towards the generator, the input admittance, ~~an~~ a parallel combination of conductance and capacitance. a maximum conductance and so on, and the cycle repeats itself repeats $\lambda/2$. When the line is traversed from the point of maximum (or Minimum) conductance to that of minimum (or Maximum) conductance. obviously, there will be a point at which the real part of the admittance is equal to the characteristic admittance.

If a suitable susceptance, obtained by using an appropriate length of a short circuited or open circuited line is added in shunt at this point so as to obtain anti-resonance with the susceptance already existing, ~~part of it~~ then upto that point, matching has been achieved.

Through there is a mismatch existing between this point and the load the effects of mismatch over this short length are in applicable. However, it is desirable that the stub to be located as near the load as possible. Also, the characteristic admittance of the stub so connected in shunt should be the same as that of the main line.

The point impedance at any point of a transmission line is given by

$$Z_{IN} = Z_0 \frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l} \longrightarrow (1)$$

converting impedance into admittance, we get

$$Y_{IN} = Y_0 \frac{Y_L + Y_0 \tanh \gamma l}{Y_0 + Y_L \tanh \gamma l} \longrightarrow (2)$$

for a high frequency line $\alpha = 0$, so that $\gamma = j\beta$. Also changing admittance into normalised admittance in the above eqn, we get

$$y_{in} = \frac{y_l + j\tan\beta l}{1 + j y_l \tan\beta l} \longrightarrow (3)$$

where $y_{in} = \frac{Y_{IN}}{Y_0}$ = normalised input admittance

and $y_l = \frac{Y_L}{Y_0}$ = normalised load admittance

Rationalising eqn(3), we get

$$y_{in} = \frac{y_l + j\tan\beta l}{1 + j y_l \tan\beta l} \times \frac{(1 - j y_l \tan\beta l)}{(1 - j y_l \tan\beta l)}$$

$$= \frac{y_l(1 + \tan^2\beta l) + j(1 - y_l^2)\tan\beta l}{(1 + y_l^2 \tan^2\beta l)}$$

for no reflection

$$Y_{IN} = Y_0 \quad (or) \quad \frac{Y_{IN}}{Y_0} = 1$$

$$\text{i.e.,} \quad y_{in} = 1$$

Thus, the stub has to be located at a point where the real part of y_{in} is equal to unity: (or)

$$\frac{y_l(1 + \tan^2\beta l_s)}{1 + y_l^2 \tan^2\beta l_s} = 1 \Rightarrow \tan^2\beta l_s (y_l - y_l^2) = 1 - y_l$$

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$$Y_L \tan^2 \beta l_s = 1 \Rightarrow \boxed{\tan\beta l_s = \frac{1}{\sqrt{Y_0}} = \sqrt{\frac{Y_0}{Y_L}}} \longrightarrow (4)$$

This equation gives the location of the stub l_s and can further be simplified as follows:

$$\beta l_s = \tan^{-1}\left(\sqrt{\frac{Y_0}{Y_L}}\right)$$

$$\frac{2\pi}{\lambda} l_s = \tan^{-1}\left(\sqrt{\frac{Z_L}{Z_0}}\right)$$

Therefore $$\boxed{l_s = \frac{\lambda}{2\pi} \tan^{-1}\left(\sqrt{Z_L/Z_0}\right)} \longrightarrow (5)$$

The susceptance at the location of the stub will be

$$\frac{b_s}{Y_s} = \frac{(1-Y_L^2)\tan\beta l_s}{1+Y_L^2 \tan^2\beta l_s}$$

Putting the value of $\tan\beta l_s$ from eqn (4), we get

$$\frac{b_s}{Y_0} = \frac{\left(1-\frac{Y_L^2}{Y_0^2}\right)\sqrt{\frac{Y_0}{Y_L}}}{1+\frac{Y_L^2}{Y_0^2}\cdot\frac{Y_0}{Y_L}} = \frac{\left(1-\frac{Y_L^2}{Y_0^2}\right)\sqrt{\frac{Y_0}{Y_L}}}{1+\frac{Y_L}{Y_0}}$$

$$= \frac{\left(1-\frac{Y_L}{Y_0}\right)\sqrt{\frac{Y_0}{Y_L}}\cancel{\left(1+\frac{Y_L}{Y_0}\right)}}{\cancel{\left(1+\frac{Y_L}{Y_0}\right)}} = \left(1-\frac{Y_L}{Y_0}\right)\sqrt{\frac{Y_0}{Y_L}}$$

$$\therefore b_s = \frac{Y_0 - Y_L}{\cancel{Y_0}} \cdot \cancel{Y_0} \sqrt{\frac{Y_0}{Y_L}}$$

$$\boxed{b_s = (Y_0 - Y_L)\sqrt{\frac{Y_0}{Y_L}}} \longrightarrow (6)$$

The short circuit is invariably used because

(i) it radiates less power and

(ii) its effective length may be varied by means of a shorting bar which normally takes the shapes of shorting plugs.

The susceptance of lossless short circuited stub is $-Y_0 \cot\beta l_t$, where l_t the length of the short-circuited stub.

Therefore, $\frac{Y_0 - Y_L}{Y_0} \cdot Y_0 \sqrt{\frac{Y_0}{Y_L}} = Y_0 \cot \beta l_t$

$$\cot \beta l_t = (Y_0 - Y_L) \sqrt{\frac{1}{Y_0 Y_L}}$$

$$\Rightarrow \beta l_t = \tan^{-1} \left[\frac{\sqrt{Z_L Z_0}}{(Z_L - Z_0)} \right]$$

$$\boxed{l_t = \frac{\lambda}{2\pi} \tan^{-1} \left(\frac{\sqrt{Z_L Z_0}}{Z_L - Z_0} \right)}$$

Disadvantages of single stub:-

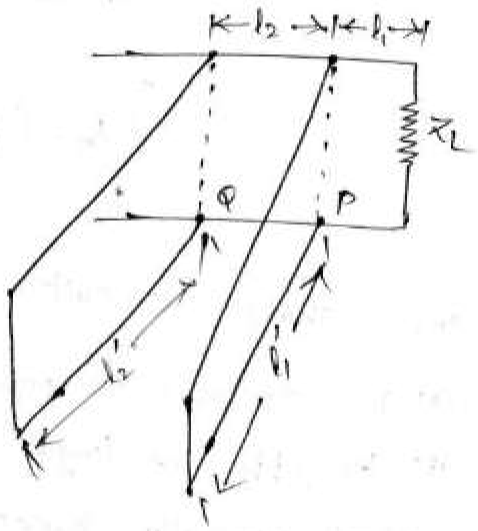
→ Disadvantages of single stub matching device are

(i) it is useful only for a fixed frequency since any frequency change requires the location of the stub to be changed.

(ii) The matching is achieved by final adjustment of the stub by moving along the line slightly. This may be suitable for open wire lines but in coaxial lines it could be inaccurate.

Double stub Matching:—

Here two short circuit stubs whose lengths are adjustable independently but whose positions are fixed. Let the first short circuited stub whose length is l_1' be located at point p at a distance of l_1 from the load end. The normalized input admittance at that point is given by



$$y_p = \frac{Y_P}{Y_0} = \frac{y_1 + j\tan\beta l_1}{1 + j y_1 \tan\beta l_1}$$

$$= \frac{y_1 + j\tan\beta l_1}{1 + j y_1 \tan\beta l_1} \times \frac{1 - j y_1 \tan\beta l_1}{1 - j y_1 \tan\beta l_1}$$

$$= \frac{y_1 (1 + \tan^2\beta l_1) + j(1 - y_1^2)\tan\beta l_1}{1 + y_1^2 \tan^2\beta l_1}$$

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$$= \frac{y_1 \sec^2 \beta l_1 + j(1-y_1^2)\tan\beta l_1}{1+y_1^2\tan^2\beta l_1}$$

$$= g_p + j b_p$$

where $g_p = \frac{y_1 \sec^2\beta l_1}{1+y_1^2\tan^2\beta l_1}$ and $b_p = \frac{(1-y_1^2)\tan\beta l_1}{1+y_1^2\tan^2\beta l_1}$

When a stub having susceptance b_1 is added at this point the new admittance will be

$$y_p' = g_p' + j b_p$$

The conductance part remains the same as only the susceptance rate gets affected due to addition of the stub. y_p' must be of such value that the admittance Y_Q equals $1+jb_p$. The stub length at Q is so adjusted that the new value of Y_Q is 1 to provide proper matching. The distance l_2 is always kept $< \lambda/2$. Generally it is chosen as either $\frac{\lambda}{4}$ or $3\lambda/8$. There is a similar restriction on length l_1 also. The total distance l_1+l_2 should be kept as small as possible to avoid reflection loss occuring to the right of 'Q' or atleast be kept at the minimum. For this reason, the first stub itself should be placed at the load. In practice, 'l_1' is generally 0.1λ to 0.15λ.

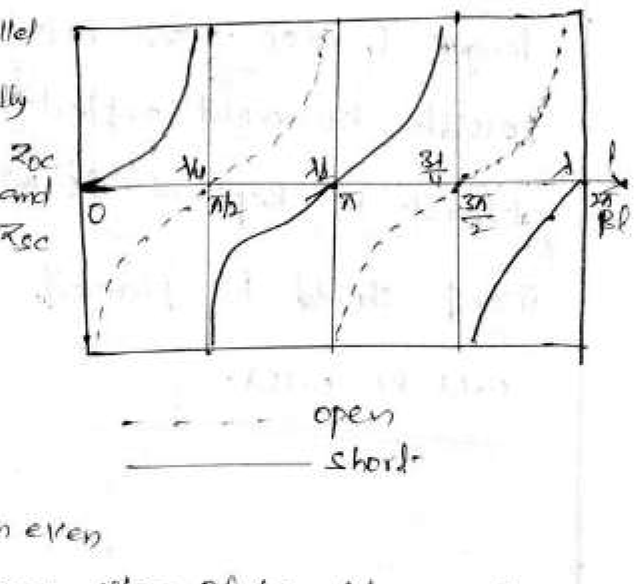
Transmission Lines as Circuit Elements:

→ for a Lossless line, $\alpha = 0$ and $\gamma = j\beta$. We know that the expression for input impedance of an open circuit and short circuit lines are

and $Z_{oc} = jZ_0 \cot\beta l$

$Z_{sc} = jZ_0 \tan\beta l$

The above expressions shows that the input impedance of an open and short circuited lossless line is a pure reactance. Desired value of reactance is obtained by varying the electrical length βl of stubs. The graph of Z_{oc} and Z_{sc} as a function of βl as shown in figure. Dotted line shows the variations of Z_{oc} with βl or l, while regular lines shows the variation of Z_{sc} with βl or l. Above the Horizontal line shows the length of line 'l' interms of wavelengths, while below Horizontal line shows the electrical length of the βl interms of π and Above the horizontal line the value of Z_{oc} and Z_{sc} will have positive reactance that is inductive while below the horizontal line the value of Z_{oc} and Z_{sc} will have negative reactance that is capacitive.



→ considering the variation of Z_{sc} only, Parallel resonance or anti resonance with theoritically infinite impedance occurs when βl is an odd multiples of $\pi/2$ and series resonance or resonance with theoritically zero impedance occurs when βl is an even multiples of $\pi/2$.

→ considering the variation of Z_{oc} only, Parallel resonance occurs when βl is an even multiple of $\pi/2$ and series resonance occurs when βl is odd multiple of $\pi/2$.